VOLUME 2

AANKALAN

DEPARTMENT OF MATHEMATICS, HANSRAJ COLLEGE

A A N K A L A N

Department of Mathematics,

Hansraj College

Volume 2, 2021

From the Principal's Desk

Aankalan, the Annual Mathematics Journal of Hansraj College is in its second year. It gives me great joy to laud the young minds who have worked assiduously to bring it to fruition.

The Mathematics Department of Hansraj College has always had a legacy of greatness; be it in the sphere of academic performance or innovation and research. Students have always been encouraged to think out of the box and apply their learnings in the real world. They have always been given the opportunities to inculcate critical and logical thinking. To this count, I congratulate and commend the efforts of the Editorial Board to carry on the legacy and seeking to promote critical thinking and research.

I also thank the faculty advisors Dr. Preeti Dharmarha, Dr. Harjeet Arora, Ms. Amita Aggarwal, Dr. Mukund Madhav Mishra and Dr. Rakesh Batra for their constant efforts to take the Department to greater heights and mentoring young minds.

I hope that Aankalan will continue to serve as a learning platform for students and uphold the sanctity and glory of the Department!

Dr. Rama
Principal
Hansraj College,
University of Delhi

From the Head of Faculty Advisors

The response to the first issue of *Aankalan*, the Academic Publication of the Department of Mathematics, Hansraj College in 2020 from students and faculty has been overwhelming. I have relished the joy of contributing in guiding our extremely dedicated and focused Editorial Team in shaping this second issue of *Aankalan*. I thank our ever dynamic and inspiring Principal, Prof. Rama, for her consistent backing and motivation.

I congratulate all the team members: Editor-in-Chief: Utcarsh Mathur along with the Associate Editors: Adityendra Tiwari, Apurva Chauhan, Samarth Rajput and Shivam Belvanshi, Assistant Editors: Daksh Dheer and Simran Singh, who despite the trying times of pandemic, endeavoured with complete dedication and determination.

We are in the midst of a marked transformation in teaching and learning, which is evident from the recently implemented *New Education Policy* in the country. There is an emphasis on improving and innovating the curriculum and instructional approaches, which need inquiry-based learning, which in turn leads to active engagement of students with coherent and meaningful mathematical tasks where students process mathematical ideas collaboratively.

The research is clearly pointing to inquiry-based mathematics education as a stance and set of teaching strategies to actively engage in mathematics teaching and learning. The desired evolution in education suggests shedding the boundaries of limited classroom learning in order to prepare the learner for global leadership. For such changes to be possible and sustainable, a broader cultural change is needed and to meet the new challenges in any field, updating with the new trends and new knowledge is the key, which is possible only with a scientific aptitude of researching and exploring.

Excellence in academics has always been a hallmark of the Department of Mathematics at Hansraj College. Mathematicians par excellence like *Shanti Narayan ji*, *Prof. M.C. Puri*, *Prof. S. C. Arora* and *Dr. S. R. Arora*; in close association with gurus like *Dr. Harbans Lal*, *Dr. K. L. Bhatla*, *Dr. N.M. Kapoor*, *Dr. Satpal*, and *Sh. J. P. Pruthi* as guiding lights brought the Department as the most sought after for undergraduate studies, the stepping stone to one's career. Our Department has been instrumental in nurturing and creating global leaders by providing the best of classroom teaching along with co-curricular and practical experiences.

We lost two pillars of our Department, Dr. Satpal and Sh. J. P. Pruthi since the issuance of our previous issue of *Aankalan*. Our entire team pays tributes to the stalwarts, whose able guidance had a significant role in nurturing the Department and our College for many decades of their association with us.

The inputs and thorough feedback by the Advisory Editorial Board members Dr. Harjeet Arora, Mrs. Amita Aggarwal, Dr. Rakesh Batra and Dr. Mukund Madhav Mishra, aided immensely in improving the content. A big thanks to them!!

The contents of this edition have also been selected after a careful review by experienced faculty. The issue comprises of a rainbow of articles ranging from abstract concepts from analysis such as totally bounded sets, ordering in Complex Numbers and Riemann Hypothesis to beautiful applications of linear algebra and probability along with the Three Utility Problem from graph theory and mathematics in plants from bio-mathematics. Apart from these, it also includes two problems namely, 'Mapping' and 'The Windmill Process' along with a short piece on lesser-known Mathematical facts.

I am sure that this publication will greatly motivate the students in developing inquisitiveness to explore new domains, go further deep into the touched upon ideas and build their interest in research; inculcating confidence in them. It is also our endeavour in honing the presentation skills of the readers and prepare them for creating quality academic content. I am confident that this launching pad will open new arenas for the avid learners and embolden them to dig out the pearls from the vast ocean of knowledge.

Dr. Preeti Dharmarha
Associate Professor,
Department of Mathematics

EDITORIAL BOARD

Faculty Advisors

Dr. Preeti Dharmarha

Dr. Harjeet Arora

Ms. Amita Aggarwal

Dr. Mukund Madhav Mishra

Dr. Rakesh Batra

From the Editors' Pen

Aankalan, the Annual Mathematics Journal of Hansraj College was started in 2019 by a group of Math-enthusiasts, with a view to encourage external learning and inculcate research and expository skills at an undergraduate level. It aims at appreciating the significant presence of the subject in real life. We seek to go beyond the books and explore the vastness of the subject through research, exploration and innovation, something that the Department has always encouraged. As Walt Whitman eloquently puts in his 1867 poem '*When I Heard the Learn'd Astronomer*', real knowledge is what comes from exploration and experiences and this is what the Department of Mathematics at Hansraj stands by.

It is believed by many that all mathematics is concerned with is cumbersome equations and unnecessary symbols carelessly dispersed over pieces of paper. But that's far from the truth. The subject is, in our humble opinion, derived from the universal truths of the world. Every iteration has some logic to it. Perhaps it is in this logic and truth, in a perplexed and asymmetric world, that mathematicians find their peace. Though the wide and all-pervasive nature of this subject has prevented the formulation of a common definition, one can agree that mathematics is concerned with patterns, certain regularities (or irregularities) that constitute nature. Be it the cycle of time, the change of weather, movement of planets or even our own breathing, patterns can be found everywhere in life. It is these patterns that form the basis of mathematics.

The Journal encourages students to explore such patterns and try to make them relatable with topics of higher mathematics. It serves as a learning platform, where students learn from each other, through the aggregation of each others' thoughts.

The initial part of the Journal brings to you the palate of the average math aficionado, with topics from Real and Complex Analysis, Abstract Algebra and Linear Algebra. These articles discuss the certain nuances and subtleties of these subjects, from the perspective of undergraduate students. We also include some clever proofs of well-known theorems.

We then move on to discuss the applications of the subject in real life. It is almost impossible to imagine a world without mathematics. It is involved in almost all aspects of our lives, be it using the internet or managing one's finances or processing data or even preparing one's food! Often, mathematics is applicable in places that we don't really see. The latter portion of this Journal deals with such observations.

Finally, we leave the reader with some food for thought; certain problems that would capture your fancy and some lesser-known facts that can be the subject of further research. We hope that the articles increase the knowledge base of students and encourage them to delve further into the particularities of the discipline. We end with a quote by someone who always valued knowledge and power of exploration;

> "*Reach high, for stars lie hidden in you. Dream deep, for every dream precedes the goal.*"

-Rabindranath Tagore

With that, we present to you, *Aankalan* 2021.

Acknowledgements

We thank our Principal, Dr. Rama, who has been a source of encouragement, motivation and inspiration for all of us and who has always supported our efforts to innovate and experiment.

A heartfelt gratitude towards Dr. S.C. Arora, for sparing time from his busy schedule and sharing with us his experiences and imparting wisdom. The Department will always follow the path illuminated by him.

We are also grateful to all the professors of the Department for their constant support and guidance. We especially thank Dr. Preeti Dharmarha, Dr. Harjeet Arora, Ms. Amita Aggarwal, Dr. Mukund Madhav Mishra and Dr. Rakesh Batra (TIC) for their thorough and comprehensive feedback of the articles, without which the Journal wouldn't have been possible.

We would like to thank the President of the Department and former Editor-in-Chief, Aman Chaudhary, who has always guided us in the right direction. Further, we thank the former team of Associate Editors; Ashutosh Maurya, Gaurav Kumar and Ishita Srivastava, for considering us worthy and capable of taking the baton and helping us in all possible ways.

We also thank the technical team comprising Mukesh Kumar, Ujjawal Agarwal, Vedant Goyal along with Dushyant Kumar Rohilla, Jivin Vaidya, Ojal Kumar and Vanshika, for working tirelessly on the Department website that gave *Aankalan* a platform. We also thank all members of the Mathematics Department Council for their constant support and encouragement.

Last, but not in anyway the least, we thank the students of the Department, for their constant efforts to make the Department reach greater heights and their overwhelming response and participation this year.

We thank you all for your support!

Interview with Dr. S.C. Arora

Prof. S.C. Arora is a teacher par excellence and an illustrious and erudite academician. He completed his post-graduation from Hansraj College, University of Delhi in 1967 standing first in the University. Subsequently, he was appointed as a lecturer in Hansraj College, where he taught for a period of twenty years. He then joined the Department of Mathematics, University of Delhi, where he also served as Head of the Department.

After superannuation, he remained associated with various academic institutions including PDM college/University, Haryana and SGTB Institute of Management and Technology, Delhi. He continues to be associated with academic bodies like UGC, AICTE, NCERT, CBSE etc. He is also associated with many Public Service Commissions like Union Public Service Commission, Himachal Pradesh Service Commission and Jammu and Kashmir Service Commission among others. Throughout his teaching career, he was involved in research and published more than 100 papers in various reputed journals and continues to deliver lectures at national and international conferences. Prof. Arora has also supervised 75 M.Phil and Ph.D students.

The representatives of the Editorial Board, **Daksh Dheer** and **Simran Singh** had a candid telephonic conversation with this calm and humble personality. Excerpts:

Daksh: Sir, first of all, we would like you to tell us about your journey at Hansraj. How was it like? What were the major highlights?
Dr Arora: I was a student here at Hansraj College in 1961, when it used to be a boys' college, and was not co-educational. Our teachers were great. In fact, it is due to their teachings, that I became Dr. S.C. Arora. The one thing that they taught us, that I would point out, was to take our classes regularly. We were always taught to be punctual and "*make no excuses*". Hence, we adjusted our time table, programs and fests in such a way that we didn't miss any classes. My teaching career at Hansraj began on 28th July, 1967. Throughout my career, I do not remember missing even a single lecture, be it during my stint at Hansraj from 1967-1987, or during my days at the University. My students can vouch for that! It isn't something worthy of praise, it is our duty to do so. If I speak in very safe terms, we are being paid to do just that!

Simran: Sir, you said you owe your teachings to your mentors, and so, we would like to know what was it like having Professor Shanti Narayan as your mentor.
Dr Arora: He was a great man. He did so much for the students. It was because of him

that people like me became faculty at the Department of Mathematics, University of Delhi. He moulded our careers. To be very frank, I was quite poor. He gave me a place here; I was living at college hostel for a year during my masters. He used to help the students so much. I still remember that during my post-graduation, I used to sit in the college library (which used to close by 12 am in those days), and Prof. Narayan used to come at 11:30 pm to see which students were studying. I must say, whatever you all must've have heard about him, that is still minuscule compared to what a great man he truly was. I distinctly remember once, I had issued a book from the library, *Commutative Algebra by Zariski*, but somehow it went out of my possession without my knowledge. I had no money to pay for the book. I went to him and explained the situation, and he wrote to the college library and waived off the fine. The years at Hansraj College were the golden years of my life. The past can never come back and I miss those days dearly, even today.

Simran: Yes, sir, hearing your anecdotes, we cannot help getting bittersweet about our own unique online college life, which brings me to ask you- what changes according to you are required in the current teaching patterns?
Dr Arora: A lot of things have changed since I retired in 2010. People have become faster now, life has become quicker. People are used to online learning, something which we never had. I do not know how useful the online mode of education is, but I do believe that the students, particularly young ones, are overburdened. Perhaps something should be done in that regard so that the students aren't always stuck to their laptops and phone screens. Thinking and introspection on one's own is also required, which is what we used to do. Some balance has to be struck between the online and offline mode. I feel care has to be taken to ensure that students learn to think deeply on their own. There is no replacement of face-to-face teaching. A teacher's personality is absorbed by their students, and not just in terms of teaching, but the way they tackle problems and talk, which is missing now. I remember talking to my students who tell me that they try and teach the way I taught them; they try to imitate my methods and teachings towards their students. So, students inherit the personality of their teachers and pass it on further to their own students.

Simran: Since we are on this topic, would you tell us how did you end up in the field of mathematics? Did you choose your profession as a professor for a specific reason?
Dr Arora: My parents never planned to send me to Hansraj College. Out of Physics, Chemistry and Mathematics, I was more inclined towards maths, even though physics was also very interesting. I simply chose mathematics because I enjoyed it. At the time, I never thought of pursuing engineering or medicine, though there weren't any competitions to qualify for going into these fields; everyone used to get admission here or there unlike now. I could have just as well gone into the field of physics or chemistry. I came here just because it happened, and I don't want to falsely boast that it was all planned and I always wanted to study mathematics- no, I simply chose it because I was better at it.

Simran: Sir, now that you have told us about choosing mathematics, could you tell us about your areas of interest? Which field do you find most interesting?
Dr Arora: My specialisation is in analysis, so I was always interested in that. However,

I used to like algebra too. Now, before I move further, I want to tell you all to not simply go by my words because now the scenario has changed. The focus has shifted to applied mathematics, i.e. people now talk more about its applications. To give you an example, some 15-20 days ago, I delivered an online lecture on the Fundamental Theorem of Algebra. I have delivered the same lecture many times, but this instance, for the first time, they insisted that I talk not entirely about the theorem, but also on its applications. Nowadays, people mostly talk about how the mathematics they study is applied to real-world problems.

Daksh: Right, sir, there is now a focus on applied mathematics now, but even so, do you think that scope for undergraduate research should be provided? How can students be encouraged in this direction?
Dr Arora: I think they may be shown the places where the knowledge they have learnt can be applied. At the undergraduate level, there is high scope of practicality and application of concepts, and this ought to be focused upon. Undergraduate study is mostly focused on mathematical tools; tools of algebra, tools of analysis, in other words, the utility of the subject. Of course, they may learn on their own terms, at their own pace, but the emphasis should be on applications and usage of mathematics. Moreover, focus on the real, pure mathematics should come at postgraduate and higher levels, when the students have attained enough maturity and understanding.

Simran: On that note sir, how do you view student-teacher bond, and what word of advice would you like to give to the students?
Dr Arora: Love and respect; they go together. Teachers show love to their students and, in turn, students show respect to their teachers. They are two inseparable aspects. The bond is quite similar to how a father cares for his son or daughter. This sacred bond is important and ought to be preserved. To all the students, I wish them all the luck and advise them to develop this love-respect bond deeply with their teachers. I emphasise here once again the necessity of attending classes regularly without fail, be it a teacher or a student. Learn to think on your own and don't let the shortcomings of online classes deter you. Remember that these are the golden years of your life that won't ever come back, so live them to the fullest and work hard!

Contents

EQUIVALENT DEFINITION OF TOTALLY BOUNDED SETS

Ekansh Jauhari
Batch of 2020

Abstract
The article aims to provide a more convenient equivalent definition of totally bounded sets in a metric space by establishing that the subset $F_\epsilon \subset X$ (in the usual definition of totally bounded sets in a metric space- also defined in the *Introduction* section below) can be chosen from the set A itself, i.e. we can get some set $G_\epsilon \subset A \subset X$ corresponding to any given $\epsilon > 0$.

Notations: (x_n) denotes sequence. $x_n^{(i)}$ denotes that *i* is fixed index and n is the running index. $U \subset V$ includes both possibilities, $U = V$ and $U \subsetneq V$. $B(x, \gamma)$ denotes open ball centered at x with radius γ.

Introduction

For a set in a metric space, the condition of total boundedness is much stronger than that of boundedness because it gives many important implications, including of course, the boundedness of the set.

In a metric space (X, d), $A \subset X$, $A \neq \phi$ is said to be totally bounded if $\forall\ \epsilon > 0, \exists$ a finite non-empty set $F_\epsilon \subset X, F_\epsilon = \{x_1, x_2, \ldots, x_n\}$ such that $A \subset \bigcup\limits_{z \in F_\epsilon} B(z, \epsilon)$ or $A \subset \bigcup\limits_{i=1}^{n} B(x_i, \epsilon)$. Amongst many implications of total boundedness, one important implication or property is the following:

Lemma. *A set is totally bounded if and only if every sequence belonging to the set is a Cauchy sequence (the technique involved in its proof deserves another such dedicated article, and is thus skipped here).*

PROOF STEP I

(See the *Abstract* section for the aim of this proof.)

Note that it is trivially true if the set A is finite, or if $A = X$. So we take the general case where A could be a proper subset of X and it need not be finite.

Let us be given a metric space (X, d) and $A \subset X$ be totally bounded. Choose an arbitrary $\epsilon > 0$. So, $\exists\, F_\epsilon = \{x_1, x_2, \ldots, x_n\} \subset X$ such that $A \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)$. Take some $y \in A$. Then, for some j, $1 \leq j \leq n, y \in B(x_j, \epsilon)$. Then $d(x_j, y) < \epsilon$. Let $r = \epsilon + d(x_j, y)$.

Claim 1: $B(x_j, \epsilon) \subset B(y, r)$.

Proof. $z \in B(x_j, \epsilon) \Rightarrow d(x_j, z) < \epsilon\; d(y, z) \leq d(y, x_j) + d(x_j, z)$
$\Rightarrow d(y, z) - d(y, x_j) \leq d(x_j, z)$. But we also have $d(x_j, z) < \epsilon$
$\Rightarrow d(y, z) - d(y, x_j) < \epsilon$ or $d(y, z) < d(y, x_j) + \epsilon = r$
Therefore $z \in B(y, r)$ and therefore $B(x_j, \epsilon) \subset B(y, r)$. □

So, what we get from here is that for arbitrary $y \in A, y \in B(x_j, \epsilon)$ for some j, $1 \leq j \leq n$ and that $B(x_j, \epsilon) \subset B(y, \epsilon + d(x_j, y)) \Rightarrow y \in B(y, \epsilon + d(x_j, y))$. Let us denote this $y \in B(x_i, \epsilon)$ by $y_i \in B(x_i, \epsilon)$.

We will get these $y_i \,\forall\, 1 \leq i \leq n$. Clearly, as $y_i \in A \,\forall\, 1 \leq i \leq n$, we get a new set $G_\epsilon = \{y_1, y_2, \ldots, y_n\} \subset A$ and (rewriting with the changed notation), we have $A \subset \bigcup_{i=1}^{n} B(y_i, \epsilon + d(x_i, y_i))$ for the initially chosen $\epsilon > 0$, where y_i are chosen from A such that $y_i \in B(x_i, \epsilon)$. If we define $M = sup\{d(x_i, y_i) : 1 \leq i \leq n\}$, then certainly $M \in \mathbb{R}^+$ and clearly, $B(y_i, \epsilon + d(x_i, y_i)) \subset B(y_i, \epsilon + M) \,\forall\, 1 \leq i \leq n$.

Thus, finally we get that for any given $\epsilon > 0$, we can get a finite set $G_\epsilon \subset A, G_\epsilon = \{y_1, y_2, \ldots, y_n\}$ such that $A \subset \bigcup_{i=1}^{n} B(y_i, \epsilon + M)$ where $M = sup\{d(x_i, y_i) : 1 \leq i \leq n\}$, and $y_i \in B(x_i, \epsilon)$. □

INFERENCES

For a given $\epsilon > 0$, we're able to form a finite totally bounding cover from within A, but of radius $\epsilon + M$. So to get a cleaner form, for any given $\epsilon > 0$, we will have to perform the following 3 steps:

1. Choose an $\epsilon > 0$ and get a conventional total bounding cover from $F_\epsilon = \{x_1, x_2, \ldots, x_n\} \subset X$.

2. Find (one for each i from 1 to n) $y_i \in A,\ y_i \in B(x_i, \epsilon)$ and calculate $M = sup\{d(x_i, y_i) : 1 \leq i \leq n\}$.

3. Finally, make the total bounding cover $G_\epsilon = \{y_1, y_2, \ldots, y_n\} \subset A$ with radius $\epsilon - M$, which finally gives us the radius $(\epsilon - M) + M = \epsilon$.

Note 1: the value $\epsilon - M$ is justified as a non-negative radius because $d(x_j, y_j) < \epsilon\ \forall\ j, 1 \leq j \leq n$ whereas $M = sup\{d(x_i, y_j) : 1 \leq i \leq n\}$. So by definition of supremum, $M \leq \epsilon$.

Though the proof is easy, the actual procedure to get G_ϵ is unnecessarily lengthy. We can get a better result if we can directly form a total bounding of the same radius $\epsilon > 0$ just after step 1. Proving this shall require slightly different arguments and use of the above lemma (see 1).

Proof Step II

In continuation with the previous arguments for (possibly infinite) totally bounded $A \subset X$, and given $\epsilon > 0,\ A \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)$. Consider a $y_1 \in A$ such that for some $j, 1 \leq j \leq n,\ y_1 \in B(x_j, \epsilon)$.

Now, let T_c denote $\bigcup_{a=1}^{c} B(y_a, \epsilon)$, and thus $T_1 = B(y_1, \epsilon)$.

If $(B(x_j, \epsilon) \cap A) \subset T_1$, we are done. So assume otherwise. Then, $(B(x_j, \epsilon) \cap A) \setminus T_1 \neq \phi, \Rightarrow \exists\ y_2 \in (B(x_j, \epsilon) \cap A) \setminus T_1$. Now, make $T_2 = B(y_1, \epsilon) \cup B(y_2, \epsilon)$. If $(B(x_j, \epsilon) \cap A) \subset T_2$, we are done, otherwise $\exists\ y_3 \in (B(x_j, \epsilon) \cap A) \setminus T_2$ and we form T_3 and perform similar check again.

Note 2: Evidently, $y_2 \notin B(y_1, \epsilon)$ and $y_3 \notin B(y_1, \epsilon), B(y_2, \epsilon)$ and thus, $d(y_2, y_1) \geq \epsilon$ and $d(y_3, y_2) \geq \epsilon,\ d(y_3, y_1) \geq \epsilon$.

Claim 2: $\exists\ c \in \mathbb{N}$ such that $(B(x_j, \epsilon) \cap A) \subset T_c$.

Proof. Let us prove this by the method of contradiction and assume that $\forall\ n \in \mathbb{N}, (B(x_j, \epsilon) \cap A) \setminus T_n \neq \phi$, and therefore $\exists\ y_{n+1} \in (B(x_j, \epsilon) \cap A) \setminus T_n$.

Clearly then, $\forall\ n \in \mathbb{N}, y_{n+1} \notin \bigcup_{a=1}^{n} B(y_a, \epsilon)$.

$\Rightarrow y_{n+1} \notin B(y_a, \epsilon)\ \forall\ a, 1 \leq a \leq n$

$\Rightarrow d(y_{n+1}, y_a) \geq \epsilon\ \forall\ a, 1 \leq a \leq n$

Since this happens $\forall\ n \in \mathbb{N}$, in light of **Note 2**, we get that $d(y_e, y_p) \geq \epsilon\ \forall\ e, p \in \mathbb{N}$ where $e \neq p$. Consider the sequence of these terms $(y_e) \in A\ \forall\ e \in \mathbb{N}$. This sequence $(y_e) \in A$ is a Cauchy sequence if and only if $\forall\ \delta > 0, \exists\ n_0 \in \mathbb{N}$, such that $\forall\ e, p \geq n_0,\ d(y_e, y_p) < \delta$.

But from above, we obtained, for a particular, (initially chosen for the proof) $\epsilon > 0$ and $\forall\, n \in \mathbb{N}$, that $d(\, y_e, y_p) \, \geq \epsilon \,\forall\, e, p \in \mathbb{N}$ where $e \neq p$.

This, of course, is an absolute negation to the definition of Cauchyness which proves that $(\, y_e) \, \in A$ is NOT a Cauchy sequence. But then this entire derivation is a clear contradiction to the result (lemma 1) seen above $\Rightarrow$ the assumption that we took for this proof is wrong, and hence we get that $\exists$ some $c \in \mathbb{N}$ such that $(\, B(\, x_j, \epsilon) \, \cap A) \, \subset T_c$. □

This $c \in \mathbb{N}$ and these $\{y_1, y_2, \ldots, y_c\}$ are for the particular ball $B(\, x_j, \epsilon)$, so better we denote them as $c_j \in \mathbb{N}$ and $\{y_1^{(j)}, y_2^{(j)}, \ldots, y_{c_j}^{(j)}\}$. As the choice of $B(\, x_j, \epsilon)$ was arbitrary, we can get some c_i and the corresponding set of points $\{y_1^{(i)}, y_2^{(i)}, \ldots, y_{c_i}^{(i)}\} \,\forall\, i, 1 \leq i \leq n$.

Therefore, $\forall\, 1 \leq i \leq n, (\, B(\, x_i, \epsilon) \cap A) \, \subset T_{c_i}$ for some $c_i \in \mathbb{N}$, where $T_{c_i} = \bigcup_{a_i=1}^{c_i} B(\, y_{a_i}^{(i)}, \epsilon)$.

Now taking set union on both sides wrt *i*, $\bigcup_{i=1}^{n} (\, B(\, x_i, \epsilon) \, \cap A) \, \subset \bigcup_{i=1}^{n} (\, \bigcup_{a_i=1}^{c_i} B(\, y_{a_i}^{(i)}, \epsilon) \,) \Rightarrow$ $A \subset \bigcup_{i=1}^{n} (\, \bigcup_{a_i=1}^{c_i} B(\, y_{a_i}^{(i)}, \epsilon) \,)$

Note that $y_{a_i}^{(i)} \in A \,\forall\, 1 \leq a_i \leq c_i, \,\forall\, 1 \leq i \leq n$.

Therefore, we have found a set, say, $H_\epsilon \subset A$, such that $A \subset \bigcup_{z \in H_\epsilon} B(\, z, \epsilon)$, where $H_\epsilon = \{y_1^{(1)}, y_2^{(1)}, \ldots, y_{c_1}^{(1)}, y_1^{(2)}, y_2^{(2)}, \ldots, y_{c_2}^{(2)}, \ldots\ldots, y_1^{(n)}, y_2^{(n)}, \ldots, y_{c_n}^{(n)}\}$, which is certainly a finite subset of A of cardinality $\prod_{i=1}^{n} c_i$. As our $\epsilon > 0$ was arbitrary, this can be done for any given $\epsilon > 0$.

So, given that $A \subset X$ is a totally bounded set, we proved that $\forall\, \epsilon > 0, \exists$ a non-empty finite set $H_\epsilon \subset A$, obtained using its initial covering set $F_\epsilon \subset X$, such that $A \subset \bigcup_{z \in H_\epsilon} B(\, z, \epsilon)$. □

BIBLIOGRAPHY

[1] Notes by *Dr. Subiman Kundu*, Professor, Department of Mathematics (MZ 185), Indian Institute of Technology Delhi, Delhi.

[2] Sutherland, W. A. (1975), *Introduction to metric and topological spaces*, Oxford University Press, ISBN: 0-19-853161-3

PLAUSIBLE ORDERINGS OF THE COMPLEX PLANE

Aman Chaudhary
Year III

Abstract

The utility of complex numbers has been fascinating since its evolution with the help of roots of negative unity. Repeated attempts of ordering the complex numbers in the Argand Plane has been a topic of subtle interest to several mathematicians, unlike the real numbers where a trivial ordering is easily observable. The prime aim of the paper is to illustrate the idea of ordering and find out if there actually exists any, for the complex numbers. The paper deals with bringing out not only the conventional orderings like the lexicographic ordering or Pseudo-ordering as discussed in various standard books, but also an ordering arising through the idea of stereographic projection and its interpretation.

Keywords: *Ordering relation, Lexicographic ordering, Pseudo-ordering, Stereographic projection*

Introduction

The idea of complex numbers is believed to have first struck the head of *Hero of Alexandria* in 1^{st} century CE, referring to the square roots of negative numbers, as per evidences[5]. The idea of complex numbers is however credited to the Italian Mathematician *G. Cardano* (in 1545). Many renowned mathematicians have also put forth great interest in this idea and it proliferated with the support of mathematical minds like *R. Descartes* (coined the word *'imaginary'*), *De-Moivre*, *Euler* and many more[2].

Since the beginning, when *Argand* suggested to represent the complex numbers in the plane, proceeding with the evolution of *Order Theory*, the complex numbers' inability to be an ordered field, has put to glory the name of many mathematicians who have contributed to this field and has always fascinated a freshman in mathematics. It is a very noticeable question in nearly every standard book meant for undergrads, that 'The field of complex numbers is not a completely ordered field'. The article is somewhat inspired from this question, talking in detail not only about, why it isn't a completely ordered field, but also what it actually is, and is there any ordering possible, if yes then which one?

Section 3, dealing with the central objective of the paper, uses a very salient feature of complex numbers, i.e. stereographic projection. It initiates to provide a method that is more valuable and handy in giving an analogous ordering in $\mathbb{C}$, using the idea of equivalence relation and equi-radii complex numbers. The article is kept basic without using intense characteristics of Complex analysis or Discrete mathematics, to allow the freshmen to feel handy at it.

The key work is partitioned into three sections. Section 1 deals with the preliminaries for the article from the definition point of view along with the brief discussion of the order theory and notations used, so that the paper can be followed on smoothly. Section 2 discusses in detail about the inability of complex numbers to become an ordered field, along with addressing some common ordering relations as discussed in various textbooks and their possible reasons for failure. Section 3 represents the chief idea of the paper in connecting the stereographic projection with a special type of pre-ordering relation in the complex plane, and giving some geometrical interpretation. Section 4 concludes the paper.

PRELIMINARIES

The article is specifically centered at the ordering relation, so let's start with some formal definitions of the protagonist viz. ordered sets and ordering relations.

Partial ordering of a set, often simply referred to as ordering, is a binary relation which is reflexive, antisymmetric and transitive. A set having such a relation is called a 'poset' (Partial Ordered Set), as was first coined by *Garrett Birkhoff* [4]. A poset is said to be totally ordered (linear order or chain) if the relation holds for all pairs of elements. Since the article is concerned with the idea of $\mathbb{R}$ and $\mathbb{C}$ only, we can simplify the definition of an ordering relation as: Consider a set A, then a relation on A is said to be an ordering relation, denoted by $<$ (read as: 'is less than'), if it satisfies the following two properties:

- If $x, y \in A$ then either of the following holds

$$x < y \qquad or \qquad x = y \qquad or \qquad x > y$$

- If $x, y, z \in A$ then

$$x < y \;\; and \;\; y < z \Rightarrow x < z$$

In the same light, a set A is said to be an ordered set, if it holds an ordering relation. In the article further, ordering refers only to the total ordering in the set. The ordering relation is of extreme utility in structures like Fields. A field F is said to be an ordered field if it is an ordered set, i.e. the following four axioms (order axioms) are satisfied for any $x, y, z \in F$.

Axiom 1. *Exactly one of* $x < y, \ \ x = y \ \ or \ \ x > y$ *holds true ('Law of Trichotomy').*

Axiom 2. *If* $x, y, z \in F$ *then* $x < y \ \ and \ \ y < z \Rightarrow x < z$ *('Law of Transitivity').*

Axiom 3. *If* $x < y$ *then* $\forall \ \ z \in F, x + z < y + z.$

Axiom 4. *If* $x > 0$ *and* $y > 0$*, then* $xy > 0$.

Some common examples satisfying these properties are the set of real numbers $\mathbb{R}$ and rationals $\mathbb{Q}$. Also, preorder (denoted as $\lhd$) is a binary relation that is both reflexive and transitive, but not necessarily antisymmetric. Each preorder on a set A induces an order relation by defining an equivalence relation $\preceq$, where $a \preceq b$ if $a \lhd b$ and $b \lhd a$ and the so obtained set of equivalence class is often denoted by $A/\lhd$.

Pertaining to ordering relation in the article, $x \leq y$ denotes '$x < y \ or \ x = y$'. Also $x < y$ and $y > x$ denotes the same thing. Often, $x > 0$ and $x < 0$ will imply as being positive and negative respectively. $Re(z)$ and $Im(z)$ denotes the real part and imaginary part respectively, of a complex number. The article uses the standard notation of z for the complex number, and $|z|$ for the modulus of z (length of the line segment joining the complex number z to the origin in the Argand plane), $arg(z)$ denotes the principal argument of the complex number (angle measured from the positive real axis to the line segment joining the origin and z).

IS IT POSSIBLE TO ORDER THE COMPLEX PLANE?

Mathematicians have made repeated attempts to order the complex plane, but all their efforts in vain, as the answer is (as expected) 'NO'. Let's look at why it's not possible to order the complex plane and some possible reasons later on. But before that, a trivial lemma is mentioned below (proof as an exercise to readers), which will help us establish the fact that we can't order the complex plane.

Lemma. *If* $x > 0 \Rightarrow -x < 0$

Now, let's begin proving, using the concept of proof by contradiction.

Theorem. *There doesn't exist any ordering relation in the field of complex numbers.*

Proof. Let (if possible), $<$ be an ordering relation in the field $\mathbb{C}$. Since this ordering relation on $\mathbb{R}$ (a subset of $\mathbb{C}$) satisfies $-1 < 0$, so the ordering on $\mathbb{C}$ must also hold this. But being an ordering relation, $<$ should be able to order any two elements of the complex plane, so consider the ordering of i and 0. Since $i \neq 0$, so by *Axiom 1*, either of the following cases must hold:

Case 1: Let $0 < i$, then on multiplying by i both sides
$0 < i^2 \Rightarrow 0 < -1$ (By *Axiom 4*)

which is a contradiction.

Case 2: Now if $0 > i$, then by *Lemma*, $0 < -i$.
On multiplying by '$-i$' both sides, we have
$0 < (-i)^2 \Rightarrow 0 < -1$ (By *Axiom 4*)

which is a contradiction again.

Since it is a contradiction to a pre-established ordered pair, so our assumption was wrong. Thus, it's not possible to order 1 and i, i.e. the two elements are incomparable. Similarly, it is clear that there are infinite number of such pairs that can't be compared. Therefore, no such complete ordering exists, and so "$\mathbb{C}$ is not an ordered field".

□

Since, we now know that there is no complete ordering possible, one may further muzzle his/her head and ask, "Are there any other types of ordering possible in $\mathbb{C}$? If yes, what are they ?"

Clearly, $\mathbb{C}$ doesn't feature any complete ordering relation. But, the authors have tried making $\mathbb{C}$ an ordered field either by some sort of extension or relaxation in the conditions. For instance, Apostol[1] has mentioned in his Exercise 1.36 (Page no. 29) that is another ordering style, which does seem to be a proper ordering at once but it fails to satisfy the order axioms. This *'Pseudo-Ordering'* is defined as:
We say $z_1 < z_2$, if either of the following holds–

1. $|z_1| < |z_2|$
2. $|z_1| = |z_2|\ and\ arg(z_1) < arg(z_2)$

As the name suggests, it is not a complete ordering property but it does help to disentangle the complex plane a bit. Upon verification, reader may observe that out of the four order axioms, all the axioms hold (an exercise to reader), except *Axiom 2*. Consider $z_1 = -1$ and $z_2 = 1$, then clearly $z_2 < z_1$, as $|z_1| = |z_2|$ and $arg(z_1) = \pi > 0 = arg(z_2)$. If $z_3 = -i$, then $z_1 + z_3 = -1 - i > 1 - i = z_2 + z_3$ as $|z_1 + z_3| = |z_2 + z_3| = \sqrt{2}$, and $arg(z_1 + z_3) = \theta_1(say) = \frac{-3\pi}{4}$, while $arg(z_2 + z_3) = \theta_2(say) = \frac{-\pi}{4}$. But clearly, $|\theta_1| > |\theta_2|$ contradicts *Axiom 2*.

Apart from the Pseudo-ordering by Apostol, W. Rudin[6] has mentioned Lexicographic ordering as follows:
We say $z_1 = a + ib < z_2 = c + id$, if either of the following holds–

1. $a < c$
2. $a = c\ and\ b < d$

Though this may satisfy the order axioms, but its futility in terms of applications can be easily inferred from the fact that it is also called as 'Dictionary ordering' or 'Alphabetical ordering', whereby we just order as per the appearance of alphabets in a dictionary.

Existence of a pre-ordering in $\mathbb{C}$

The ordering that is going to be defined here uses a key feature of 'Stereographic projection', so let's first discuss what exactly it is, and how can we deploy it to come up with an ordering.

The Stereographic Projection:

Unlike real numbers, complex numbers are represented by points in a plane, since they possess two co-ordinates instead of one. The idea of representing complex numbers in the plane is believed to have been first noted by *Gauss* and *Argand* independently, but the major contributions were made by *B. Riemann*, who made the use of a sphere to project its points on a plane.

Consider a sphere of diameter 1 and centre $(0, 0, \frac{1}{2})$ in a $3-$D plane, kept such that the bottom touches the origin, and it lies in the upper half of the $z - axis$, i.e. the points lying on the plane can be represented as –

$$S = \left\{ (x, y, z) : x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4} \right\}$$

Points on the sphere are projected from the North Pole (point N lying on the $z - axis$, i.e. $(0, 0, 1)$) onto the tangent plane ($x-y$ plane) passing through the origin (South-Pole S), like point a lying on the sphere is projected to point b on the plane (see Figure 2.1). In this procedure, each point on the sphere, except the north-pole (as depicted by set S'), corresponds to exactly one point on the plane. The North-Pole is referred to as *'the only point of infinity'* in the complex plane. If set P depicts the projection of the Set S' on the $x - y$ plane, then we have:

$$S' = S \setminus \{(0, 0, 1)\}$$

$$\mathbb{C} \sim P = \{(x, y, 0) : \zeta = x + iy \in \mathbb{C}\}$$

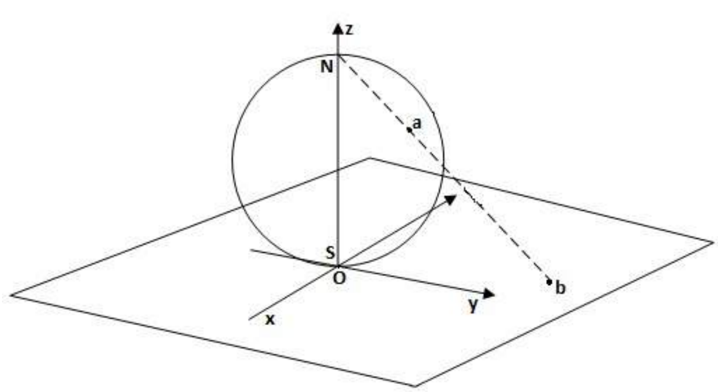


Figure 2.1: The Stereographic Projection

This phenomenon is known as stereographic projection, and the sphere is called the *'Riemann Sphere'*. Clearly, P corresponds to the complex plane while $P \cup \{0, 0, 1\}$ corresponds to the Extended Complex Plane (An extended Field that includes all the possible infinities of the Field).

Now, if we look carefully and ponder, we will realize that while projecting the points of the sphere to the complex plane, a set of infinite number of concentric circles centered at the origin $(0, 0)$ are observed, whose radii vary from 0 to $+\infty$. In this way, every complex number lies at the circumference of one of the circles with the origin as its center (the complex number $0 + i0$ can be regarded as a circle of radius 0 with center at origin).

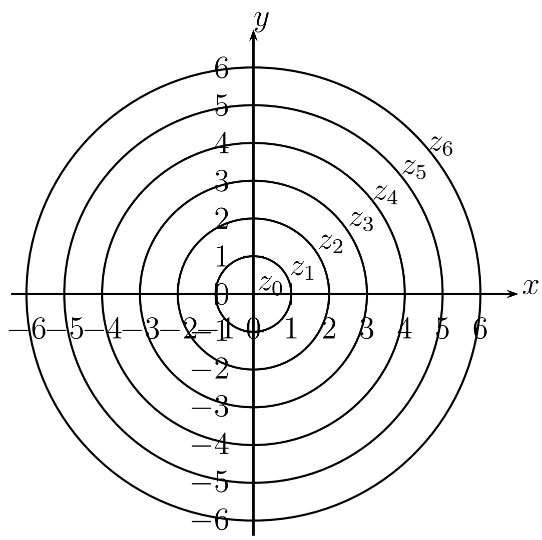


Figure 2.2: The concentric circles

So, something like Figure 2.2 is observed such that $0 = |z_0| < |z_1| < |z_2| < |z_3| < \cdots < \infty$, where $|z_i| = r_i$ and $0 \leq |r_i| < +\infty \ \forall \ i \ \in \mathbb{R}^+$, whereby the subscript of z and r denotes the radius of the circle here.

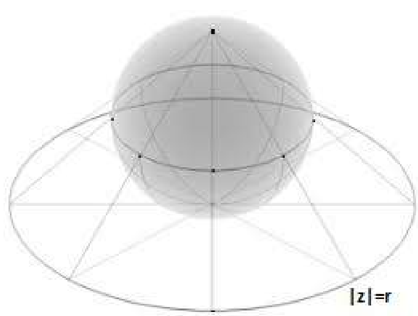


Figure 2.3: The Equi-radii complex numbers

In a nutshell, we have used the modulus of complex numbers to order them, such that the complex numbers lying on the circumference of the circles $|z_i| = r_i$ are less than those complex numbers lying on the circumference of the circles $|z_j| = r_j$, for $i < j$. But the problem remains unresolved for the complex numbers lying on the circumference of the same circle (see Figure 2.3).

For instance, all the roots of unity have the same modulus, but they are unalike, so all this theory shatters at the very point, and concludes that the ordering is not complete. But, this allows us to define a pre-ordering by using the equivalence classes of the complex numbers induced by the equivalence modulus relation, i.e. all the complex numbers lying on the circumference of the same circle are equi-radii complex numbers and they are equal in a manner that their modulus values are equal and they belong to the same class, represented as $[z_1] = [z_2]$, where [ ] denotes the equivalence class, which is analogous to saying $z_1 = z_2$.

So, we have used the law of trichotomy of real numbers to relate with a feature of complex number, which is also real, and this feature allows us to partition the complex plane in classes with the help of the equivalence modulus relation.

Clearly, for any two complex numbers z_1 and z_2, exactly one of $z_1 < z_2$, $z_1 > z_2$, or $[z_1] = [z_2]$ holds as either of $|z_1| < |z_2|$, $|z_1| > |z_2|$, or $|z_1| = |z_2|$ holds in a complete ordering. So by the above property, we are able to partition complex numbers in classes that can be ordered easily. For example, if we have to order the complex numbers $a = i,\ b = -i,\ \ c = 1+i,\ \ and\ \ d = 2-i$. Since $|a| = 1,\ |b| = 1,\ |c| = \sqrt{2}\ \ and\ \ |d| = \sqrt{5}$, so we have $|a| = |b| < |c| < |d|$ implying $[a] = [b] < [c] < [d]$, where a and b are equi-radii complex numbers.

Geometrically speaking, an ordering relation gives an idea of the position of the elements when placed over a line (linear ordering), for instance in $\mathbb{R}$, $x < y$ if x lies on the left of y, or y lies towards the right of x. Here also, this ordering via modulus relation corresponds to the analogous feature as the point on the circumference, whose modulus is greater than the other, is farther from the origin of the complex plane while the points which are equidistant from the origin of the circular plane, belong to the same equivalence class. The same property can be attributed to the areas of the circle constituted by the circumference they lie upon, i.e. if the area of the circle on whose boundary the point lies, is greater than that of the other point, then it is farther, while if the points lie on the same circle then they have the same area and therefore belong to the same equivalence class. Similarly, a property concerning the volume of the sphere bounded by the plane $x - y$ plane and a plane parallel to the $x - y$ plane, passing through the pre-image of the complex number on the sphere, can be felt by the readers.

This idea of pre-ordering has comparatively more potential in the sense that it allows us to have some properties that are prevalent in the real numbers like *Linear ordering, Archimedean property, well ordering principle* and many more. So in one way or the other, we can say that the set $\mathbb{C}$ of complex numbers is an ordered field under the equivalence classes of the modulus relation, or we can say that complex numbers is a pre-ordered field. A very relative question that may strike the reader's mind be: "Why is the ordering in $\mathbb{C}$ pretty different than that in $\mathbb{R}$?". This can be attributed to the fact that the structure of Real and complex is not exactly the same, for instance $\mathbb{R}$ is believed to have two infinities namely $+\infty$ and $-\infty$, while $\mathbb{C}$ posses only North Pole as its infinity.

CONCLUSIONS

The paper examines the idea of ordering the complex plane rigorously, in the light of Order theory and Complex analysis. The paper gives a brief idea about ordering relations, posets, partial ordering of complex plane and the complete ordering, citing instances from books where authors have tried to extend the idea of ordering to make $\mathbb{C}$ an ordered field. Discussing the pseudo-ordering and lexicographic ordering, it begets a way to pre-order the complex plane using the sense of stereographic projection and its interpretation.

BIBLIOGRAPHY

[1] Apostol Tom M., *Mathematical Analysis*, 2nd Edition, 1974, Addison-Wesley Publishing company, ISBN 0-201-00288-4

[2] Burton David M.,*The History of Mathematics*, 3rd Edition, 1995, New York: McGraw-Hill, ISBN 978-0-07-009465-9

[3] Brown; Churchill, *Complex Variables and Applications*, 6th Edition, 1995, New York: McGraw-Hill, ISBN 978-0-07-912147-9

[4] Garrett Birkhoff, Lattice Theory, American Mathematical Society Colloquium Publications, vol. 25, revised edition, 1948, American Mathematical Society, New York

[5] Nahin Paul J., *An Imaginary Tale: The Story of* $\sqrt{-1}$, 1998, Princeton University Press, ISBN 978-0-691-12798-9

[6] Rudin Walter, *Principles of Mathematical Analysis*, 3rd Edition, 1976, McGraw-Hill Education, ISBN 978-0070856134

[7] Spiegel, Lipschutz and others, *Complex Variables*, 2nd Edition, 2009, Schaum's Outline Series, McGraw Hill, ISBN 978-0-07-161569-3

[8] Yadav D.K., A New Approach To Ordering Complex Numbers, Int. J. of Mathematical Sciences and Engg. Applications, Vol.2, No.3, 211-223 (2008), ISBN 0973-9424, Pune, India

RIEMANN HYPOTHESIS: THE PRIME QUESTION

Utcarsh Mathur
Year II

Abstract

One of the biggest landmarks in Mathematical history was perhaps created with the publication of Georg Friedrich Bernhard Riemann's 1859 paper titled '*On the Number of Primes Less Than a Given Magnitude*', presented to the Berlin Academy. The paper had many claims that brought about changes in the study of Number Theory, Complex Analysis and most importantly theory of Prime Numbers. The nature of primes continues to be an enigma in Mathematics that is yet to be understood. It was the Riemann Hypothesis that gave us some direction. This paper discusses the Riemann-Zeta Function in brevity and attempts to explain its consequences on the Prime Number Theorem.

Keywords: *Analytic Number Theory, Riemann-Zeta Function, Prime Number Theorem*

A Brief Introduction

The concept of prime numbers is one that is taught to a student at a very young age, perhaps in the sixth grade itself. Ironically, however, a lot is yet to be known about the nature of prime numbers. After a point, it is difficult to understand their occurrence and even more difficult is to interpret their density.

This paper explains the correlation between the Riemann-Zeta function and the Distribution of Primes as theorized by Bernhard Riemann in his 1859 paper. Although numerous mathematicians such as *Eratosthenes*, *Hadamard*, *Poussin*, *Gauss*, *Hardy* have extensively studied primes, the most significant contributors to the Theory of Prime Numbers are *Euclid* (described their infinite nature), *Euler* (gave the Euler Product, discussed in a later section) and *Riemann*[1].

Through the zeroes of the Riemann-Zeta function (sometimes also referred to as the Euler-Riemann-Zeta function), Riemann gave his own prime counting function. However, to understand the essence of this function, certain preliminaries must be established.

The Riemann-Zeta function is a function of an argument s where $s \in \mathbb{C}$. It is defined as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

The argument $s = x + iy$ such that $Re(s) > 1$ (i.e. Real part is greater than one).

Lemma. $\zeta(s)$ *is absolutely convergent* $\forall\, Re(s) > 1$, *but divergent otherwise.*

Having clearly defined the nature of the function for $Re(s) > 1$, we can move on to understand its nature outside the half-plane $Re(s) > 1$. It is apparent that $\zeta(s)$ becomes divergent for any $Re(s) \leq 1$.

However, to study the relation between the function and prime numbers, the function should be defined in the whole of the complex plane. This may be possible if we define the function a little differently for $Re(s) \leq 1$, while ensuring that its basic properties are preserved. This is precisely what Riemann did in his paper. This process is called analytic continuation, although Riemann does not mention this process explicitly, rather tries to find a function which "*remains valid for all* s"[2]. The process of analytic continuation tries to extend the definition of a function beyond its domain while preserving its basic properties such that the extended function has a derivative everywhere.

Through analytic continuation, Riemann extended the definition of the function $\forall\, Re(s) > 0$ except for $s = 1$, where there is a singularity. He further extends the function in the entire complex plane using the functional equation:

$$\xi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{3.1}$$

where

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$$

defined as the gamma function.

Riemann Hypothesis

Having established the preliminaries, one can move on to explain the subject matter at hand; Riemann's Hypothesis. It can clearly be seen in 3.1 that $\xi(s)$ becomes 0 when $s = 2n$. When n is positive, the zeroes get cancelled by the poles of the gamma function. Thus,

$$\zeta(s) = 0 \;\forall\, s = -2n$$

where $n \in \mathbb{Z}^+$.

These are called the trivial zeroes of the Zeta function. In his 1859 paper, Riemann hypothesised that the non-trivial zeroes of the zeta function lie in the critical strip (region where $0 < Re(s) < 1$) and have their real part as $\frac{1}{2}$. Thus, $s = \frac{1}{2} + xi$, are the only non-trivial zeroes of the zeta function. This is called the Riemann Hypothesis and is yet to be proved[3].

Euler's Identity

In his thesis titled '*Various Observations about Infinite Series*' (published in 1737), Euler gave his product identity that gave a connection between prime numbers and the Riemann-Zeta function. This identity, called Euler's Product is one of the most important results in Analytic Number Theory and also served as the basis of Riemann's 1859 paper. The identity is:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}. \tag{3.2}$$

where the product on the right hand side is taken over all the primes.

The above identity may be derived by successively multiplying $\zeta(s)$ by $\frac{1}{p^s}$ and subtracting the resultant from the preceding term. For example:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{3.3}$$

Multiplying by $\frac{1}{2^s}$ on both sides:

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots \tag{3.4}$$

Subtracting 3.4 from 3.3 we get

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots \tag{3.5}$$

In a similar manner, one may successively multiply the Zeta function by a reciprocal of some power of a prime number and subtract the term obtained from the preceding term. Thus,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}$$

By looking at the identity 3.2 one may also realise that the Riemann-Zeta function cannot be zero for any $Re(s) > 1$. Thus, the only zeroes $\zeta(s)$ has are:

1. Its trivial zeroes i.e. every negative even integer.
2. Its non-trivial zeroes given by the critical line $s = \frac{1}{2} + xi$ in the Argand plane (as claimed by Riemann).

RIEMANN'S PRIME COUNTING FUNCTION

The Prime Counting function $\pi(x)$ is defined as the number of primes less than a given number x. For example, $\pi(3) = 1$ as there is only one prime number less than 3. When graphed, this function resembles the step-function as it increases by one unit for each prime. While it is easy to compute $\pi(x)$ for smaller values of x, this cannot be done for arbitrarily large numbers.

A number of other methods have been proposed by various mathematicians to compute the density of prime numbers, the most accurate being the one given by Riemann. He defines his Prime Counting function as:

$$J(x) = Li(x) - \sum_{\rho}^{\infty} Li(x^{\rho}) - \log 2 + \int_{x}^{\infty} \frac{1}{t(t^2-1)\log t} dt \tag{3.6}$$

where

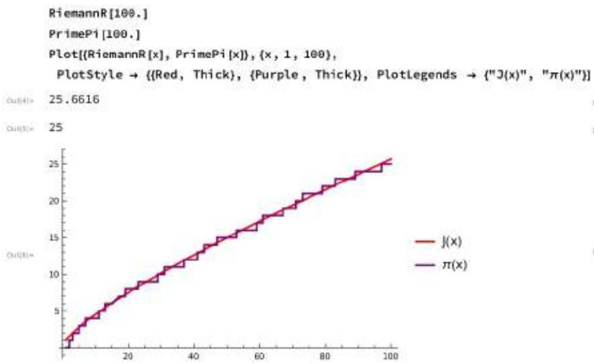
$$Li(x) = \int_{2}^{x} \frac{1}{\ln t} dt$$

and ρ represents the non-trivial zeroes of the Riemann-Zeta function[1]. Riemann claimed in his paper that the above function could calculate the number of primes less than a given magnitude with the least error.

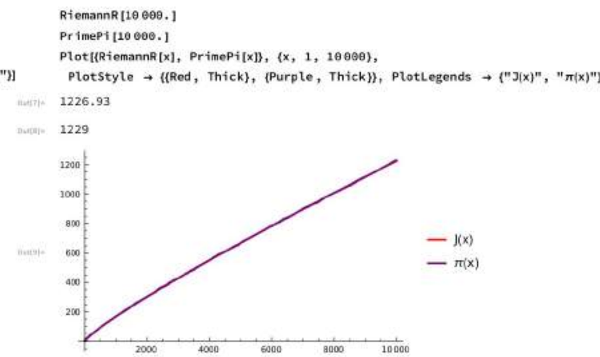
The accuracy of the Riemann-Prime Counting function may be appreciated with the following examples. Here, the Prime Counting function and Riemann's Counting Function have been calculated and graphed (using CAS *Wolfram Mathematica*, due to their large nature) to provide a comparison.

The following commands have been used in the figures below:

1. **RiemannR[ ]**: for calculating Riemann's Prime Counting Function.
2. **PrimePi[ ]**: for computing the Prime Counting Function.
3. **Plot[ ]**: for plotting the functions mentioned above.

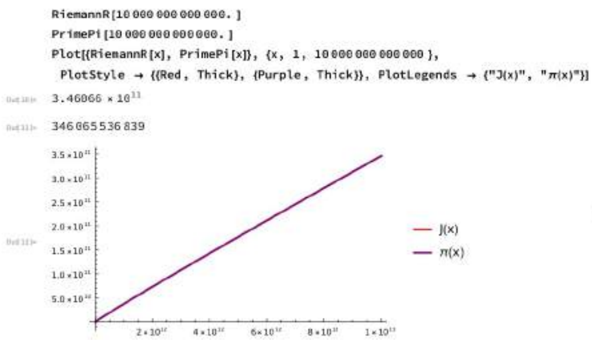


(a) $\pi(x)$ and $J(x)$ for $x = 100$

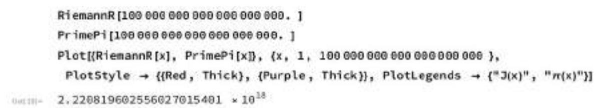


(b) $\pi(x)$ and $J(x)$ for $x = 10,000$

Figure 3.1



(a) $\pi(x)$ and $J(x)$ for $x = 10^{13}$



(b) $\pi(x)$ and $J(x)$ for $x = 10^{20}$

Figure 3.2

Remark. In the figures, the following observations can be made:

1. $\pi(x) = J(x) + \kappa$ where κ is the error. Thus, $J(x) \approx \pi(x)$
2. The error term increases gradually as x increases.
3. After a point, even the CAS fails to give an accurate value for $\pi(x)$, forcing one to rely on $J(x)$.

Thus, when x is arbitrarily large, no conclusions can be drawn from the Prime Counting function. Riemann's function (accounted for error) is the only way to get a rough picture. Even though this function has given credible results, it can only be considered true when it is proved that the non-trivial zeroes of the Riemann-Zeta function (denoted by ρ in his Prime Counting function) have $\frac{1}{2}$ as their real part.

Conclusions

The above observations highlight the relevance of the Riemann-Zeta function in modern Mathematics. Though his Prime Counting function has approximated prime number density with most accurately vis-à-vis other functions, its validity solely depends on the Hypothesis. Riemann himself remarked

> "One would, of course, like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation."[5]

If solved, the Riemann Hypothesis would not only make us understand the true nature of Prime Numbers, but would also vindicate countless theorems that assume it to be true. Consequently, the Theory of Riemann-Zeta function has emerged as a fascinating branch of study. The Riemann Hypothesis has been checked for the first 10^{13} values but has not been proved as yet. It is enlisted as one of the 7 Millennium Problems in Mathematics by the Clay Mathematics Institute[4].

In September 2018, British-Lebanese mathematician Sir Michael Atiyah claimed to have solved the problem. His proof has been derived from the works of John von Neumann and Friedrich Hirzebruch. However, it is yet to be verified.

BIBLIOGRAPHY

[1] Derbyshire,J. 2003. *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. Washington, DC, ISBN 978-0-309-51257-2

[2] Edwards,M.H.1974. *Riemann's Zeta function*. New York, Academic Press, ISBN 978-0-486-41740-0

[3] Jørgen Veisdal: Cantor's Paradise: The Riemann Hypothesis, explained

[4] Clay Mathematics Institute: Riemann's 1859 Manuscript

[5] Riemann,B.*On the Number of Prime Numbers less than a Given Quantity* Translated by Wilkins,David R. December 1998

THE PERFECT DICE

Adityendra Tiwari
Year II

Abstract

It has become more of a norm, that the level of sophistication achieved by a civilization determines how progressive it is. This mundanity starts with someone's ingenious observation of a very grassroot subject, which is followed by the employment of our available knowledge and resources to generalise this observation and apply this newly acquired wisdom in some other field. But, there are 'rare' cases when an object of discovery is left undisturbed in its pristine form. One of these rarities is the subject of focus in this paper; a pair of dice. When numbered from 1 to 6, the probability of obtaining any particular sum with these dice is $\frac{1}{36}$ for a 2, $\frac{2}{36}$ for a 3, and so on. On the other hand, if we number one die with integers $1, 2, 2, 3, 3, 4$ and another die with $1, 3, 4, 5, 6, 8$ then the probability of obtaining any particular sum with these dice remains the same! This paper attempts to ensure its validity and prove that this is the only possible pair with such property.

Keywords: *Probability, Group Theory, Ring Theory, Integral Domains, Unique Factorisation Domain*

Introduction

The game of Ludo has deep-seated roots in the history of our country. Have you ever wondered why we have numbers only from 1 to 6 on the die *only*? And even after playing this game for more than 5000 years, this game does not seem to have reached any level of sophistication. Ludo is a game of probabilities and the uncertainty lies on the part of the die. Thinking about probability an American Mathematician, Martin Gardner presented before us, the *Sicherman Dice*[1]. It consists of a pair of dice, one with the integers $1, 2, 2, 3, 3, 4$ and the other with $1, 3, 4, 5, 6, 8$ as its labels. The astounding beauty of these dice lies in their property of yielding the same probability to achieve a given sum as that with an ordinary dice. To understand the proof better, we equip our readers with a few key concepts in mathematics that are used in this paper.

What is a Ring?

A ring *R* is a set together with two binary operations, addition and multiplication, which satisfy the following axioms:

First, *R* is an abelian group under addition with zero as identity. Next, multiplication is closed and associative. At last, we require addition and multiplication are compatible i.e. $\forall\ a, b, c \in R,\ (a+b)c = ac + bc$.

A ring $(R, +, \cdot)$ is said to be a ring with zero divisors if for any non-zero $a \in R,\ \exists$ a non-zero element $b \in R:\ ab = 0$ or $ba = 0$, where 0 is the additive identity in *R*. Here, a and b are called divisors of zero. If e is an element of a ring *R*: $ae = a = ea\ \forall\ a \in R$, then the ring is called Ring with unity. A commutative ring with unity and no zero-divisors is called an *Integral Domain*[2, 3].

What is a Unique Factorization Domain?

Essentially, a U.F.D is an Integral Domain with an additional property. This additional property may seem to be stuff reserved especially for mathematicians, but in essence, it is easy to understand:

Any element that belongs to a U.F.D can be expressed uniquely as a product of a finite number of irreducible elements of the U.F.D. Now, with the help of the terms mentioned above, we begin with the proof.

Proof. The fact that the set of integers $\mathbb{Z}[x]$ has the unique factorization property provides us with the fuel to begin with our proof. We start with the basics and consider the possibilities of getting 6 as the sum with an ordinary pair of dice and they are: (1,5), (2,4), (3,3), (4,2), (5,1).
Next we multiply two polynomials formed by the ordinary dice labels as exponents:

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

It is important to notice that the term x^6 in this product can be obtained in precisely the following ways:

$$x^1 \times x^5, x^2 \times x^4, x^3 \times x^3, x^4 \times x^2, x^5 \times x^1.$$

There is a correspondence between pairs of labels whose sums are 6 and the pairs of terms whose products are x^6. This is a one-to-one correspondence and it's valid for all sums and all dice- including the *Sicherman dice* and any other dice that yields the desired probabilities. Let us suppose $m^1, m^2, m^3, \ldots, m^6$ and $n^1, n^2, n^3, \ldots, n^6$ be any two arrays of positive integer labels for the faces of a pair of dice with the property that the probability of rolling any particular sum with these dice (*random dice*) is same as that of ordinary dice. Using our observation about the products of polynomials this means that:

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^3 + x^4 + x^5 + x^6)$$

$$= (x^{m_1} + x^{m_2} + x^{m_3} + x^{m_4} + x^{m_5} + x^{m_6}) \times (x^{n_1} + x^{n_2} + x^{n_3} + x^{n_4} + x^{n_5} + x^{n_6}) \quad (4.1)$$

Now we have to find $m_i's$ and $n_i's$ by solving this equation[1]. Here is where the unique factorization property of $\mathbb{Z}[x]$ helps us.

The polynomial $x + x^2 + x^3 + x^4 + x^5 + x^6$ has the factorization into irreducibles as $x(x + 1)(x^2 - x + 1)(x^2 + x + 1)$. Hence, the L.H.S. of Equation 4.1 has the factorization: $x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2$.

So, this means that these factors are the only possible irreducible factors of $P(x) = x^{m_1} + x^{m_2} + x^{m_3} + x^{m_4} + x^{m_5} + x^{m_6}$. Thus, $P(x)$ has the form $x^a(x+1)^b(x^2-x+1)^c(x^2+x+1)^d$, where $0 \leq a, b, c, d \leq 2$. To narrow down on further possibilities for these four parameters, we evaluate $P(1)$ in given two ways:

$$P(1) = 1^{m_1} + 1^{m_2} + \cdots + 1^{m_6} = 6.$$

$$\text{and } P(1) = 1^a 2^b 3^c 1^d.$$

This means that b=1 and c=1. Evaluating $P(0)$ in two ways shows that:

$$P(0) = 0^{m_1} + 0^{m_2} + \cdots + 0^{m_6} = 0.$$

$$\text{and } P(0) = 0^a 1^b 1^c 1^d.$$

Thus, $a \neq 0$. If we take a=2, the smallest possible sum one could obtain with the *random dice* would be 3. Since this is in contradiction with our assumption, the only possibilities left for a, b, c, d are a=1, b=1, c=1, and d=0,1,2.

For, d=0, $P(x) = x + x^2 + x^2 + x^3 + x^3 + x^4$, hence, die labels are $1, 2, 2, 3, 3, 4$ a *Sicherman die.*
For d=1, $P(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$, so, die labels are $1, 2, 3, 4, 5, 6$ an ordinary die.
For d=2, $P(x) = x + x^3 + x^4 + x^5 + x^6 + x^8$, so, die labels are $1, 3, 4, 5, 6, 8$ the other *Sicherman die.* □

The *Cayley's Table* for both the pair of dice is as under:

$\oplus$	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Table 1: Ordinary Dice

$\oplus$	1	2	2	3	3	4
1	2	3	3	4	4	5
3	4	5	5	6	6	7
4	5	6	6	7	7	8
5	6	7	7	8	8	9
6	7	8	8	9	9	10
8	9	10	10	11	11	12

Table 2: Sicherman Dice

Conclusions

This proves that the *Sicherman dice* does give the same probabilities as the ordinary dice and that it is the only other pair of dice that has this property.

BIBLIOGRAPHY

[1] Gallian A. Joseph, *Contemporary Abstract Algebra*, 8th Edition, Boston, 2010, ISBN 978-81-7391-269-2

[2] Rings-Introduction, Educational Media Research Centre (EMRC), Madurai Kamaraj University

[3] Class of Rings, Educational Media Research Centre (EMRC), Madurai Kamaraj University

MATHEMATICS AND THE RUBIK'S CUBE

Samarth Rajput
Year II

Abstract

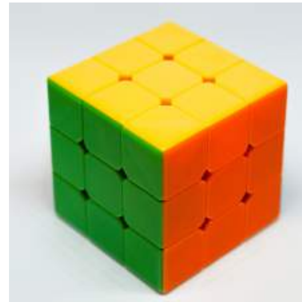
Rubik's Cube is a very famous 3-D puzzle invented by Hungarian sculptor and professor of architecture *Ernö Rubik* in 1974. The toy has been in the market for a long time and is still capable of capturing a good sale, which shows the popularity and uniqueness of this simple-looking toy. Mathematicians have shown a lot of interest in the Rubik's Cube but they didn't stop right after solving it. They looked further for the hidden mathematical logic in the cube. After doing a lot of research, they were able to link the plaything with the concept and theorems of abstract algebra. This paper is concerned with the permutation groups of a Rubik's Cube and how they are applicable in search of the maximum number of possible combinations of a Rubik's Cube that can be solved. In this paper, we'll be dealing only with a 3×3 Rubik's Cube.

Introduction

A 3×3 Rubik's Cube consists of 6 faces and each one of them is differently coloured. The combination of the cube can be changed by rotating the faces of the cube. Each one of the 6 faces is composed of 9 facets and on each face the center facet is fixed and can't be moved. In total, there are 54 facets ($9 \times 6 = 54$) and to solve a Rubik's Cube all the 9 facets of a face must be of the same colour.

A 3×3 Rubik's Cube is made up of 26 cubies. There are 6 center cubies, 8 corner cubies and 12 edge cubies. Center cubies are fixed with 1 facet each, as mentioned earlier. A corner cubie has 3 facets each and an edge cubie consists of 2 facets each. When a Rubik's Cube is rotated, only the edge and corner cubies change their positions in such a way that they move on to some other edge and corner respectively. The faces of the cube can be rotated by $90°, 180°, 270°$ in clockwise or counter-clockwise direction. A $90°$ turn is considered as 1 move and a $180°$ turn makes it 2 moves, and so on[3].

(a) A puzzled 3×3 Rubik's Cube: All the 9 facets of a face are not of the same colour.



(b) A solved 3×3 Rubik's Cube: All the 9 facets of each face are of the same colour i.e., each face is of one colour only.

Figure 5.1

Each move makes a lot of positional changes in the cube. Now, let's say we have a 3×3 Rubik's Cube. Consider a group G of transformations of the Rubik's Cube. Before moving ahead, let's label the different facets of the Rubik's Cube and number the non-center facets from 1 to 48 to see that a Rubik's Group can be regarded as a permutation of the numbers $1, 2, 3, 4, \ldots, 48$, thus forming a symmetric group S_{48}. This will also give us an idea of how a turn changes the positions of the facets. Let:

U = Upper Face of the cube
L = Left Face of the cube
R = Right Face of the cube
D = Downward Face of the cube
F = Front Face of the cube
B = Backward Face of the cube

Note: All the above face labels denote the $90°$ rotation of the respective faces in the clockwise direction.

On rotating the upper face by $90°$, the edge cubies of U get permuted by $4 \rightarrow 2 \rightarrow 5 \rightarrow 7 \rightarrow 4$ and similarly, corner cubies of U are also permuted. Cubies of other faces are also permuted along with the cubies of U. The permutation of the cubies of U can be written as disjoint 4-*cycles.* The disjoint cycle of edge cubies of U is $(4, 2, 5, 7)$. 4-*cycles* are odd permutations. For example, let's consider the 4-*cycle* $(4, 2, 5, 7)$, it can be written as $(4, 2)(2, 5)(5, 7)$, as we can see, there are three transpositions of U which ensure that 4-*cycles* are odd permutations. But, the product of two odd permutations is even, the product of two even permutations is also even and the product of an even and an odd permutation is odd and each face turn is an even permutation of the cubies as each face turn is a composition of 4-*cycles* on the corners and 4-*cycles* on edges.

U=(4, 2, 5, 7)(1, 3, 8, 6)(9, 33, 25,17)(10, 34 ,26, 18)(11, 35 , 27 ,19)

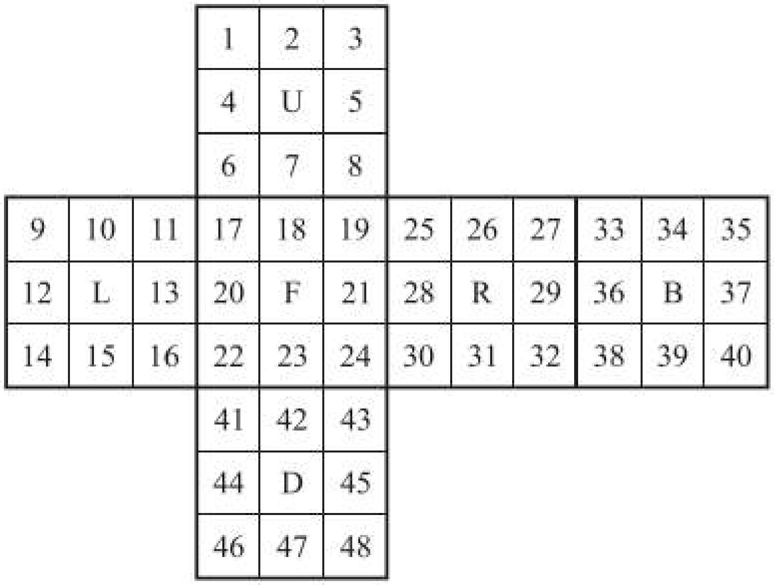


Figure 5.2: Cubie Labelling Showing $G < S_{48}$[4]

The above permutations are the result of a $90°$ turn of U in the clockwise direction. We know that, U is a product of disjoint 4-*cycles*, which means that

$$U^{-1} = U^3$$

Possible permutations for the other 5 faces of the cube:

L = $(9, 11, 16, 14)(10, 13, 15, 12)(1, 17, 25, 17)(4, 20, 44, 37)(6, 22, 46, 35)$

R = $(17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11)$

F = $(25, 27, 32, 30)(26, 29, 31, 28)(3, 38, 43, 19)(5, 36, 45, 21)(8, 33, 48, 24)$

B = $(33, 35, 40, 38)(34, 37, 39, 36)(3, 9, 46, 32)(2, 12, 47, 29)(1, 14, 48, 27)$

D = $(41, 43, 48, 46)(42, 45, 47, 44)(14, 22, 30, 38)(15, 23, 31, 39)(16, 24, 32, 40)$

Any combination in the Rubik's Cube can be described as a permutation from the solved state, which means that G is a subgroup of a permutation group of 48 elements. Here, we have formed a specific group of Rubik's Cube, G = $< U, L, R, F, B, D >$ which is a subgroup of S_{48}.

Order of the group

Now, we know about permutations, and we'll proceed to calculate the order of the Rubik's Cube group or maximum possible combinations of a Rubik's Cube from where we can bring it to a solved state. Suppose, we have dismantled a Rubik's Cube and all the edge and corner pieces are out but the center pieces are at their positions. We have 8 corner pieces and 12 edge pieces. Now, we'll notice that there are 8 corner positions and any of the 8 corner pieces can be placed at any of them. While placing the first corner piece, we have 8 positions and it can be placed at any one of them, after placing 1 corner piece, we have 7 corner places to position the second piece and so on. In the same way, edge pieces can be placed at 12 different positions.

Total possible positions of edge pieces = $12!$
Total possible positions of corner pieces = $8!$

As mentioned earlier, each edge piece is made up of 2 colours, so any of the 12 edge pieces can be placed in a cube in 2 different ways or simply, by flipping the colours which means we have twice as many ways to put those edge pieces at 12 different places. Similarly, any of the 8 corner pieces can be placed in 3 different ways, as each corner piece is made up of 3 different colors.

No. of ways in which edge pieces can be placed (Edge Flips) = 2^{12}
No. of ways in which corner pieces can be placed (Corner Twists) = 3^8

Maximum possible combinations = $12! \times 8! \times 2^{12} \times 3^8$

However, not all of these are possible to solve. It's impossible to have one and only one edge flipped which means only half of the combinations are solvable. Similarly, only one corner can't be twisted and the rest of the cube solved, which brings down the number to two-third of the number produced after removing half of the impossible cases produced due to the flipping of a single edge piece. It is also not possible to have only 2 pieces switched because each move is the composition of 4-*cycles* on the corners and the edges, and as discussed earlier, the product of two odd permutations is even. Only even permutations are allowed. Only half of the permutations are even. Thus, maximum possible combinations:

$$\frac{12! \times 8! \times 2^{12} \times 3^8}{2 \times 2 \times 3} = 43,252,003,274,489,856,000$$

[2]

Conclusions

The number of possible combinations of a 3×3 Rubik's Cube that can be solved are

$$43,252,003,274,489,856,000$$

and this is known only because of the application of mathematics. There are a lot of other applications of the permutation groups too which can be useful in solving a scrambled Rubik's Cube, but are out of bounds of this paper.

BIBLIOGRAPHY

[1] Gallian A. Joseph, *Contemporary Abstract Algebra*, 8thEdition, Boston, 2010, ISBN 978-81-7391-269-2

[2] Article. Group Theory and the Rubik's Cube

[3] Wikipedia- Rubik's Cube Group

[4] Article. The Man Who Found God's Number

[5] Article. Group Theory and the Rubik's Cube

ON SOLVING A CUBIC EQUATION

Daksh Dheer
Year I

ABSTRACT
In the 19th century, *Évariste Galois* proved that all algebraic equations of degree higher than 4 cannot always be solved by radicals, i.e. there does not exist a formula that relates the roots of the equations to their coefficients using the operations of addition, multiplication, subtraction, division, exponentiation and taking n^{th} roots. However, equations of degree 1, 2, 3 and 4 are solvable by radicals. This article attempts to introduce one such formula; specifically, one that can solve a cubic equation.

Keywords: *Cubic Formula, Cubic Discriminant, Depressed cubic, Cardano's formula*

INTRODUCTION

In algebra, an equation of the form $ax^3 + bx^2 + cx + d = 0$ such that $a, b, c, d \in \mathbb{R}$ is called a cubic equation in one variable. The solutions of the equation are called its *roots*[6]. By the Fundamental Theorem of Algebra and the fact that complex roots occur in conjugate pairs, there always exists a real root of every cubic equation. However, unlike the quadratic formula (which enables one to quickly find roots of a quadratic equation), a *cubic formula* is rarely discussed in high-school and most undergraduate courses. We attempt to derive such a formula in this article[2].

The cubic formula was first published in 1545, by *Girolamo Cardano* in his book, '*Ars Magna*'. Cardano attributed the formula to *Scipione del Ferro*. However, another mathematician, *Niccolò Tartaglia*, had also independently discovered a formula for solving cubics.

DEPRESSED FORM OF A CUBIC

All general cubics can be reduced into a *depressed cubic* of the form

$$y^3 + py + q = 0$$

which is much easier to solve, and hence, shall be the focus of this article. Reduction of a general cubic into this form can be done by a simple change of variable $x = y - \frac{b}{3a}$. This substitution is motivated by the fact that the inflection point of the general cubic has $-\frac{b}{3a}$ as its abscissa (see 6.1). The abscissa of the inflection point can be evaluated by equating the second derivative to zero. Graphically, this amounts to shifting the coordinate axes such that the inflection point now lies on the y-axis (see 6.1).

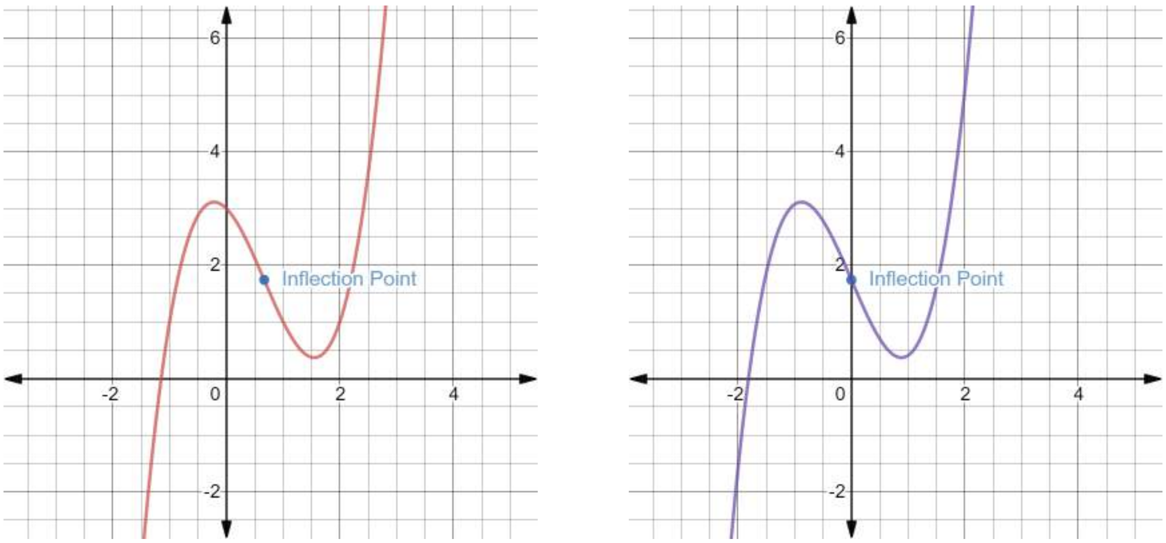


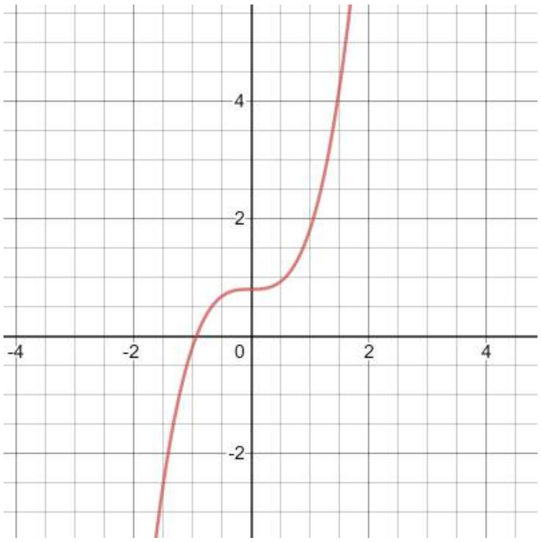
Figure 6.1: Shifting of the graphs

To revert to the original equation, one can simply use the relation $y = x + \frac{b}{3a}$. Thus, there exists a bijection between a general cubic and its depressed form. Hence, solving the depressed cubic amounts to solving every cubic.

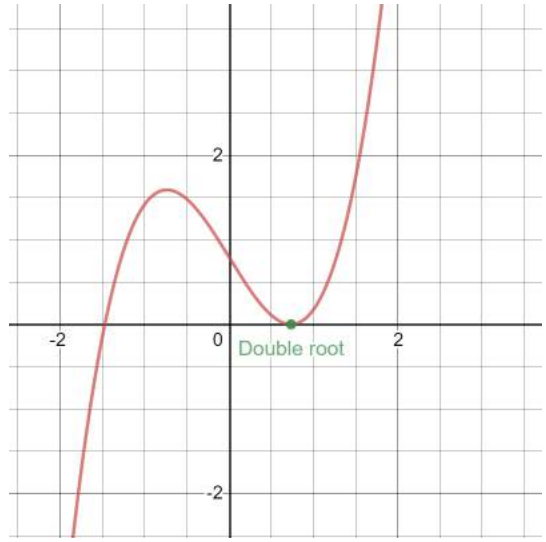
Number of solutions

We know by the Fundamental Theorem of Algebra, that every cubic has at least one real root. In this section, we shall determine a *cubic discriminant*, which will allow us to decide how many real solutions exist. Since, complex roots occur in conjugate pairs, a cubic equation has exactly one real root or exactly three real roots, not all distinct. Let us consider the first case, i.e., there exists one real root. This means that the graph of the cubic intersects the x-axis only once (see 6.2).

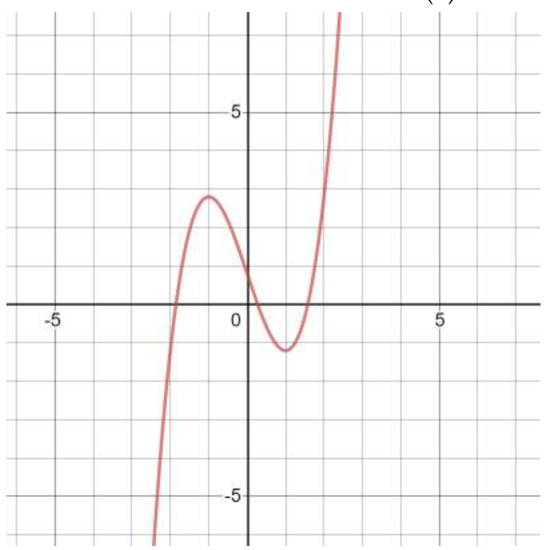
As a result, the length of the perpendicular drawn from the origin to the inflection point on the y-axis is always *greater* than the distance between the inflection point and the local maxima. Let us consider the *positive difference* between these distances. The length of the perpendicular from origin to the inflection point is simply the y-intercept of the curve, q.



(a) 1 Real Root



(b) 2 Distinct Real Roots



(c) 3 Distinct Real Roots

Figure 6.2

Now, the other distance can be calculated by taking the difference between the ordinate of the inflection point and the ordinate of the local maxima. We can get the local maxima by equating the first derivative to zero and applying the first derivative test; the local maxima occurs at the point $x = \sqrt{\frac{-p}{3}}$, and thus the ordinate is $y = \frac{2p}{3}\sqrt{\frac{-p}{3}} + q$. Hence, the distance we require is simply $\frac{2p}{3}\sqrt{\frac{-p}{3}}$.

Thus, the required difference is given by $q - \frac{2p}{3}\sqrt{\frac{-p}{3}}$, and we need to solve the inequality

$$q - \frac{2p}{3}\sqrt{\frac{-p}{3}} \; > \; 0$$

which simplifies to

$$\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 \; > \; 0$$

This is the condition for a cubic equation to have one real root.

Note that here p represents the slope of the line tangent to the inflection point, which is negative, and thus the quantity $\sqrt{\frac{-p}{3}}$ is real.

Now, let us consider the other case, i.e. there exist three real roots (not necessarily distinct). This means that the graph of the cubic intersects the x-axis thrice (see 6.2). In case of a repeated root, it shall technically intersect the x-axis twice, but for the sake of brevity, we will consider that equivalent to three intersections. In this case the length of the perpendicular drawn from the origin to the inflection point on the y-axis is always *smaller* than the distance between the inflection point and of the local maxima. Hence, we can conclude that the condition for existence of three real roots is given by

$$\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 \; \leq \; 0$$

Hence, this quantity acts as a *discriminant* for cubic equations and so we shall henceforth refer to it as Δ for ease of writing.

Completing the cube

As the counterpart to completing the square technique in the derivation of the quadratic formula, let us attempt the same in case of a cubic equation.

Consider the identity

$$(v+u)^3 = v^3 + u^3 + 3vu(v+u)$$

Re-arranging, we obtain

$$(v+u)^3 - 3vu(v+u) - (v^3+u^3) = 0$$

Comparing this with our depressed cubic, we can deduce the following relations:

$$p = -3vu$$

$$q = -(v^3+u^3)$$

$$y = v+u$$

Hence, if we obtain u and v, our original equation is a perfect cube and our equation is effectively solved.

Solving the above equations, for u and v, and subsequently adding them, we get

$$\boldsymbol{y = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt[2]{\frac{q^2}{4} + \frac{p^3}{27}}}}$$

More compactly, the same formula can be written as

$$\boldsymbol{y = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt[2]{\Delta}}}$$

The above is called *Cardano's Formula*[1]. Note that we are assuming the existence of a solution by the Fundamental Theorem of Algebra and moreover, the system of equations can be algebraically reduced to a quadratic equation in v^3. Hence, it can be solved using the quadratic formula.

A Complex Detour and an Example

One may notice the fact that the cases of one real root or three non-distinct real roots, i.e. when $\Delta > 0$ or $\Delta = 0$, respectively, correspond to cube roots of real numbers and can thus be calculated. However, in the case of existence of three distinct real roots, $\Delta < 0$ and hence we are required to take the sum of two complex numbers. This might seem contradictory and impossible; obtaining three real roots by adding up two complex numbers, but it is possible since the two complex numbers being added are conjugates of one another.

Furthermore, a cube root of a complex numbers will result in three different complex numbers and hence, we shall have three pairs of u and v. Adding corresponding values of u and v in a pair will give us the three required roots.

Now, let us solve an actual cubic equation using this method. Consider the following equation:

$$x^3 - 26x^2 + 193x - 420 = 0$$

Here, $a = 1$, $b = -26$, $c = 193$, and $d = -420$. Thus, replacing x by $y - \frac{b}{3a}$, i.e., $y + \frac{26}{3}$ and simplifying, we get:

$$y^3 - \frac{97}{3}y - \frac{1330}{27} = 0$$

Now,

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = \left(\frac{\frac{-1330}{27}}{2}\right)^2 + \left(\frac{\frac{-97}{3}}{3}\right)^3$$

$$\Delta = \frac{442225}{729} - \frac{912673}{729} = -\frac{470448}{729} \;<\; 0$$

Thus, there exist 3 real distinct roots.

Now, we know that

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt[2]{\Delta}}$$

Therefore, putting values we get:

$$y = \sqrt[3]{-\frac{\frac{-1330}{27}}{2} + \sqrt[2]{-\frac{470448}{729}}} + \sqrt[3]{-\frac{\frac{-1330}{27}}{2} - \sqrt[2]{-\frac{470448}{729}}}$$

$$y = \sqrt[3]{\frac{1330}{54} + \sqrt{\frac{44^2}{3}}\,i} + \sqrt[3]{\frac{1330}{54} - \sqrt{\frac{44^2}{3}}\,i}$$

Calculating these cube roots (one method of doing this can be first converting them to their polar forms and then carrying out the operations; other methods exist too, but the final answers remain the same), we get three different sums of conjugate pairs of complex numbers. Adding them up, we get the following values for y:

$$y = -\frac{5}{3}, \;\; -\frac{14}{3} \;\text{ and }\; \frac{19}{3}$$

Now, using the relation, $x = y - \frac{b}{3a} = y + \frac{26}{3}$, we obtain the solutions to our original equation as:

$$x = 7, \;4 \text{ and } 15$$

The General cubic formula

We have seen before that every general cubic can be reduced to a depressed cubic and we derived a formula for solving such a reduced equation[5]. However, since the mapping from general cubics to depressed cubics is bijective, we can also derive a direct formula for a general cubic as follows:

$$x = -\frac{b}{3a} + \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt[2]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt[2]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$

CONCLUSIONS

We come to the conclusion that every general cubic equation can be reduced to a depressed cubic and solved using the formula derived above. In case of three real distinct roots, the cubic formula outputs answers as sum of complex conjugates and thus, it is beyond the curfew of high school mathematics and is rarely discussed at that level, even though it is algebraically calculable.

BIBLIOGRAPHY

[1] G Cardano, *Ars Magna or The Rules of Algebra*, translated and edited by T.R. Witmer, Cambridge, MA: MIT Press 1968 and New York, NY: Dover 1993, ISBN 978-0486678115.

[2] Article. Cardano v Tartaglia: The Great Feud Goes Supernatural.

[3] Osler, Thomas. (2021). *An easy look at the cubic formula.*

[4] Chahal, J.S. *Solution of the Cubic.* Reson 11, 53–61 (2006).

[5] S Neumark, *Solution of Cubic and Quartic Equations*, Oxford, etc.: Pergamon 1965, ISBN 978-0-08-011220-6.

[6] Wikipedia Cubic Equation.

APPLICATION OF LINEAR ALGEBRA IN SOLVING GPS EQUATIONS

Yash Jain
Year II

Abstract
This article is an attempt to describe the working of the Global Positioning System (GPS). The mathematical equations involved in the process of determining the position of a receiver are described. Finally, some principles of Linear Algebra are used to solve these equations and get the receiver's location.

Keywords: *GPS, Pseudorange*

Introduction

Global Positioning System (GPS) has a wide range of applications in various industries such as military, aviation, marine and agriculture. With the development of smartphone applications with GPS capabilities, many people use the GPS navigation services of these applications to find their location or to get directions to their destination. Some smartphone applications which rely on GPS services to function efficiently are Google Maps, Uber, Ola and Swiggy. But, how does the GPS find the exact location of a user? What is the role of Linear Algebra in finding that position?

Working of the GPS

The GPS is a satellite-based radio navigation system that is owned by the United States Government[2] . It uses the process of trilateration to locate a specific point on the earth. This means that it uses the distance between the receiver and at least 3 satellites to determine the receiver's position. A GPS system consists of GPS satellites orbiting the earth which transmit signals in its direction. The signal transmitted from each satellite is encoded with the 'Navigation Message,' which can be read by the user's GPS receivers. The navigation message includes orbit parameters from which the receiver can compute satellite coordinates.

These are the Cartesian Coordinates in a geocentric system which has origin at the earth centre of mass, Z-axis pointing towards the North Pole, X pointing towards the Prime Meridian, and Y at right angles to X and Z to form a right-handed orthogonal coordinate system. The time at which the signal is transmitted from the satellite is encoded on the signal. The satellites carry very stable atomic clocks to record the time. Time of signal reception is recorded by receiver using an atomic clock. The receiver measures the difference in these times. This is used to calculate the pseudorange, which is defined as[1]:

$$\text{Pseudorange= (time difference)} \times \text{(speed of light)}$$

Pseudoranges include clock errors because the receiver clocks are not perfect. Hence, including the unknown receiver clock error and the coordinates of the unknown station, we have 4 unknowns. Hence, we need atleast 4 pseudoranges and hence, signals from 4 GPS satellites[1]. Corresponding to each pseudorange, the following equation can be obtained:

Pseudorange = Range (distance between receiver and satellite) + Clock Error

Hence, we get 4 non-linear equations in 4 unknowns. This system of equations is then linearized and solved using linear algebra.

EVALUATION OF PSEUDORANGE

The method used in this section to evaluate the pseudorange is adopted from[3]. Let the location of the receiver be (X, Y, Z). Let the location of i^{th} satellite be (X_i, Y_i, Z_i), its signal travel time be Δt_i, and the range of the receiver from that satellite be R_i, where i varies from 1 to 4.

Let Δt_0 denote the clock error and Δt_{sat_i} denote the time difference measured by the receiver. Let c denote the speed of light. Then,

$$R_i = \Delta t_i.c \tag{7.1}$$
$$\Delta t_{sat_i} = \Delta t_i + \Delta t_0 \tag{7.2}$$

Let ρ_i denote pseudorange for the i^{th} satellite, where i varies from 1 to 4.

$$\rho_i = \Delta t_{sat_i}.c$$

From equation (7.2), we have:

$$\rho_i = (\Delta t_i + \Delta t_0).c$$
$$\Rightarrow \rho_i = \Delta t_i.c + \Delta t_0.c \tag{7.3}$$

Substituting Δt_i from equation (7.1) in equation (7.3), we have:

$$\rho_i = R_i + \Delta t_0.c \tag{7.4}$$

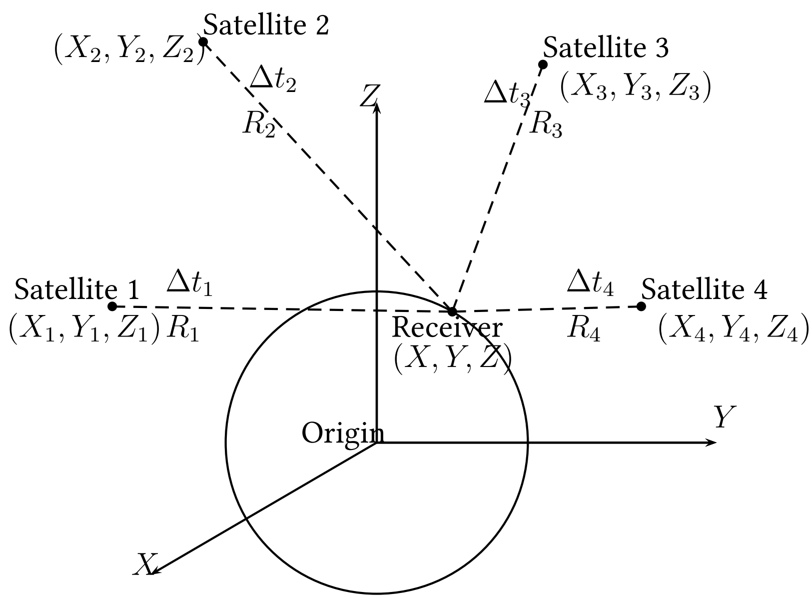


Figure 7.1

The range R_i from the satellite to the receiver can be calculated in a Cartesian system as follows:

$$R_i = \sqrt{(X_i - X)^2 + (Y_i - Y)^2 + (Z_i - Z)^2} \tag{7.5}$$

Putting the value of R_i from equation (7.5) in equation (7.4), we get:

$$\rho_i = \sqrt{(X_i - X)^2 + (Y_i - Y)^2 + (Z_i - Z)^2} + \Delta t_0.c \tag{7.6}$$

Please note that equation (7.6) is a system of 4 non-linear equations. The next step is to linearize this system of equations so that it can be solved using the principles of Linear Algebra.

Linearization of the Equations

The method used in this section to linearize the system (7.6) is adopted from[3]. According to Taylor's series,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}.(x - x_0) + \frac{f''(x_0)^2}{2!}(x - x_0) + \frac{f'''(x_0)^3}{3!}(x - x_0) + \cdots$$

Let $\Delta x = x - x_0$. Then,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}.(\Delta x) + \frac{f''(x_0)^2}{2!}(\Delta x) + \frac{f'''(x_0)^3}{3!}(\Delta x) + \cdots$$

Simplifying only first 2 terms, we get:

$$f(x) = f(x_0) + f'(x_0).\Delta x \tag{7.7}$$

In equation (7.7), x_0 is an arbitrarily estimated value about which, the function's value is estimated. We have a similar motive for R_i in equation (7.4). We have to linearize R_i so that we can estimate its value and linearize the system of 4 equations described in (7.6). Hence, we need to incorporate an arbitrarily estimated value x_0 in the vicinity of x. This means that instead of calculating (X,Y,Z) directly, we use an estimated position (X_t, Y_t, Z_t) which includes some unknown error.

Let R_{ti} denote the range of the receiver from the i^{th} satellite, where i varies from 1 to 4.

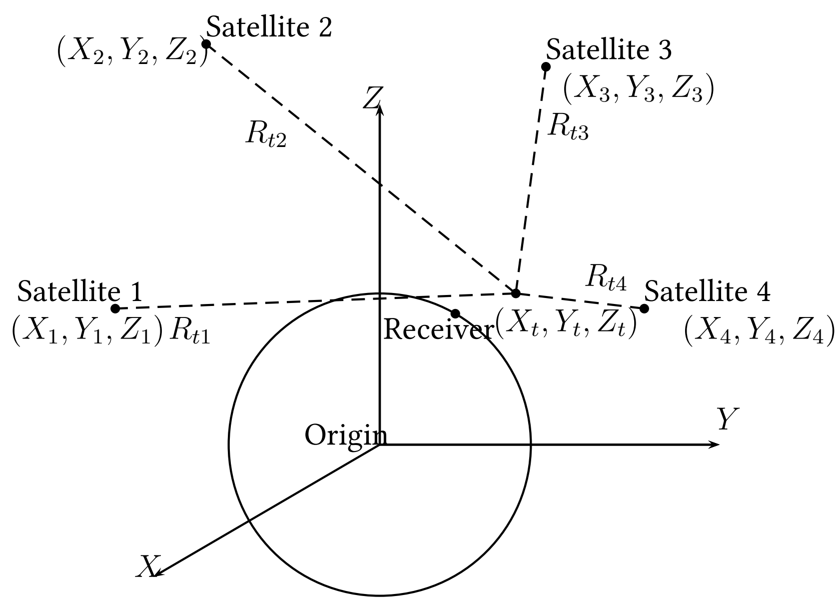


Figure 7.2

The estimated position includes an error produced by the unknown variables $\Delta x, \Delta y$ and Δz. Hence, the receiver's coordinates are estimated as:

$$\begin{aligned} X &= X_t + \Delta x \\ Y &= Y_t + \Delta y \\ Z &= Z_t + \Delta z \end{aligned} \tag{7.8}$$

The range R_{ti} from the i^{th} satellite to the estimated position can be calculated in the same way as R_i was calculated in equation (7.3):

$$R_{ti} = \sqrt{(X_i - X_t)^2 + (Y_i - Y_t)^2 + (Z_i - Z_t)^2} \tag{7.9}$$

Taking R_t from equation (7.3) as the function in equation (7.7), we get:

$$R_i = \sqrt{(X_i - X_t)^2 + (Y_i - Y_t)^2 + (Z_i - Z_t)^2} + \frac{\partial\left(\sqrt{(X_i - X_t)^2 + (Y_i - Y_t)^2 + (Z_i - Z_t)^2}\right)}{\partial x}.\Delta x$$
$$+\frac{\partial\left(\sqrt{(X_i - X_t)^2 + (Y_i - Y_t)^2 + (Z_i - Z_t)^2}\right)}{\partial y}.\Delta y + \frac{\partial\left(\sqrt{(X_i - X_t)^2 + (Y_i - Y_t)^2 + (Z_i - Z_t)^2}\right)}{\partial x}.\Delta z$$

$$\Rightarrow R_i = \sqrt{(X_i - X_t)^2 + (Y_i - Y_t)^2 + (Z_i - Z_t)^2} + \frac{-2(X_i - X_t)}{2\sqrt{(X_i - X_t)^2 + (Y_i - Y_t)^2 + (Z_i - Z_t)^2}}.\Delta x$$
$$+\frac{-2(Y_i - Y_t)}{2\sqrt{(X_i - X_t)^2 + (Y_i - Y_t)^2 + (Z_i - Z_t)^2}}.\Delta y + \frac{-2(Z_i - Z_t)}{2\sqrt{(X_i - X_t)^2 + (Y_i - Y_t)^2 + (Z_i - Z_t)^2}}.\Delta z \tag{7.10}$$

Substituting R_{ti} from equation (7.9) in equation (7.10) we get:

$$R_i = R_{ti} + \frac{X_t - X_i}{R_{ti}}.\Delta x + \frac{Y_t - Y_i}{R_{ti}}.\Delta y + \frac{Z_t - Z_i}{R_{ti}}.\Delta z \tag{7.11}$$

Now, substituting R_i from equation (7.11) in equation (7.4), we get:

$$\rho_i = R_{ti} + \frac{X_t - X_i}{R_{ti}}.\Delta x + \frac{Y_t - Y_i}{R_{ti}}.\Delta y + \frac{Z_t - Z_i}{R_{ti}}.\Delta z + c.\Delta t_0 \tag{7.12}$$

Equation (7.12) now represents a system of 4 linear equations which can be solved using Linear Algebra.

Solving the Equations

The method used in this section to solve the system (7.12) is adopted from[3]. Equation (7.12) can be rewritten as:

$$\rho_i - R_{ti} = \frac{X_t - X_i}{R_{ti}}.\Delta x + \frac{Y_t - Y_i}{R_{ti}}.\Delta y + \frac{Z_t - Z_i}{R_{ti}}.\Delta z + c.\Delta t_0 \tag{7.13}$$

The system of linear equations (7.13) has to be solved for the variables ($\Delta x, \Delta y, \Delta z$ and Δt_0). To do so, it can be expressed in the form of matrix equation $AX = B$, where:

$$
\begin{aligned}
A &= \begin{bmatrix} \frac{X_t - X_1}{R_{t1}} & \frac{Y_t - Y_1}{R_{t1}} & \frac{Z_t - Z_1}{R_{t1}} & c \\ \frac{X_t - X_2}{R_{t2}} & \frac{Y_t - Y_2}{R_{t2}} & \frac{Z_t - Z_2}{R_{t2}} & c \\ \frac{X_t - X_3}{R_{t3}} & \frac{Y_t - Y_3}{R_{t3}} & \frac{Z_t - Z_3}{R_{t3}} & c \\ \frac{X_t - X_4}{R_{t4}} & \frac{Y_t - Y_4}{R_{t4}} & \frac{Z_t - Z_4}{R_{t4}} & c \end{bmatrix} \quad (7.14) \\
X &= \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_0 \end{bmatrix} \quad (7.15) \\
B &= \begin{bmatrix} \rho_1 - R_{t1} \\ \rho_2 - R_{t2} \\ \rho_3 - R_{t3} \\ \rho_4 - R_{t4} \end{bmatrix} \quad (7.16)
\end{aligned}
$$

Using the matrices described in the above equations, $AX = B$ for the system (7.13) is:

$$
\begin{bmatrix} \frac{X_t - X_1}{R_{t1}} & \frac{Y_t - Y_1}{R_{t1}} & \frac{Z_t - Z_1}{R_{t1}} & c \\ \frac{X_t - X_2}{R_{t2}} & \frac{Y_t - Y_2}{R_{t2}} & \frac{Z_t - Z_2}{R_{t2}} & c \\ \frac{X_t - X_3}{R_{t3}} & \frac{Y_t - Y_3}{R_{t3}} & \frac{Z_t - Z_3}{R_{t3}} & c \\ \frac{X_t - X_4}{R_{t4}} & \frac{Y_t - Y_4}{R_{t4}} & \frac{Z_t - Z_4}{R_{t4}} & c \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_0 \end{bmatrix} = \begin{bmatrix} \rho_1 - R_{t1} \\ \rho_2 - R_{t2} \\ \rho_3 - R_{t3} \\ \rho_4 - R_{t4} \end{bmatrix} \quad (7.17)
$$

To solve for X, (7.17) is pre-multiplied by A^{-1} to get $X = A^{-1}B$, i.e.,

$$
\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_0 \end{bmatrix} = \begin{bmatrix} \frac{X_t - X_1}{R_{t1}} & \frac{Y_t - Y_1}{R_{t1}} & \frac{Z_t - Z_1}{R_{t1}} & c \\ \frac{X_t - X_2}{R_{t2}} & \frac{Y_t - Y_2}{R_{t2}} & \frac{Z_t - Z_2}{R_{t2}} & c \\ \frac{X_t - X_3}{R_{t3}} & \frac{Y_t - Y_3}{R_{t3}} & \frac{Z_t - Z_3}{R_{t3}} & c \\ \frac{X_t - X_4}{R_{t4}} & \frac{Y_t - Y_4}{R_{t4}} & \frac{Z_t - Z_4}{R_{t4}} & c \end{bmatrix}^{-1} \cdot \begin{bmatrix} \rho_1 - R_{t1} \\ \rho_2 - R_{t2} \\ \rho_3 - R_{t3} \\ \rho_4 - R_{t4} \end{bmatrix} \quad (7.18)
$$

The system (7.18) gives the value of the matrix X which gives the values of $\Delta x, \Delta y, \Delta z$ and Δt_0. Using the equations (7.8), we can find (X, Y, Z). However, these values still have some error (which corresponds to $\Delta x, \Delta y$ and Δz respectively).

Hence, our aim is to decrease the error component as much as possible so that we can get more accurate value of (X, Y, Z). To accomplish this, we estimate a new set of values, X_{new}, Y_{new} and Z_{new} using the calculated (X, Y, Z) values as X_{old}, Y_{old} and Z_{old} in the following equations:

$$
\begin{aligned}
X_{new} &= X_{old} + \Delta x \\
Y_{new} &= Y_{old} + \Delta y \\
Z_{new} &= Z_{old} + \Delta z \quad (7.19)
\end{aligned}
$$

where $\Delta x, \Delta y$ and Δz are unknown errors.

The new estimated position $(X_{new}, Y_{new}, Z_{new})$ is entered in place of X_t, Y_t and Z_t in the linear system (7.18) to get the values of Δx, Δy and Δz for this new estimation. The same iterative process is repeated until the error components are less than the desired error. Hence, in order to determine a position, the receiver will either use the last measurement value, or estimate a new position using (7.19) and calculate error components down to zero by repeated iteration. After we have satisfactorily low values of $\Delta x, \Delta y$ and Δz, the final position of the receiver is calculated as:

$$\begin{aligned} X &= X_{new} \\ Y &= Y_{new} \\ Z &= Z_{new} \end{aligned} \tag{7.20}$$

Hence, (7.20) gives the final value of (X, Y, Z). The calculated value of Δt_0 at this point corresponds to receiver time error and can be used to adjust the receiver clock.

Conclusions

These calculations ignore some other sources of GPS errors such as the effect of ionosphere, the effect of troposphere and multipath errors. DGPS, SBAS, A-GPS and HSGPS are some of the improved GPS which give more accurate positions. However, one can appreciate the role of Linear Algebra in the whole process of GPS positioning.

BIBLIOGRAPHY

[1] Blewitt, G., 1997.*Basics of the GPS Technique: Observation Equations. Geodetic Applications of GPS, pp.10-54.*

[2] Dixon, T.H., 1991.*An introduction to the Global Positioning System and some geological applications. Reviews of Geophysics,29(2), pp.249-276.*

[3] Zogg, Jean-Marie, 2009. *GPS: Essentials of Satellite Navigation: Compendium: Theorie and Principles of Satellite Navigation, Overview of GPS/GNSS Systems and Applications.*U-blox ISBN 9783033021396

ON ACUTENESS OF RANDOM TRIANGLES

Shivam Baurai
Year III

ABSTRACT
The problem addressed in this article relates to finding the proportion of all triangles which are acute. This problem is solved here by using concepts from elementary geometry, linear algebra and geometric probability. From the discussion on acute angles, results on right and obtuse triangles follow immediately.

The objective of the article is to consider the problem of finding the proportion of all triangles which are acute. The approach is surprisingly elementary and uses concepts from elementary geometry, linear algebra and geometrical probability. Since only the shape of a triangle is sought, and its features such as perimeter and area are irrelevant, we do not require any information more than the fact that we're working with a triangle. Furthermore, since the sides of the triangle are not considered here either, we work with only the interior angle sum.

Suppose x,y, and z are the angles of a triangle. Then, we have that $x + y + z = \pi$, where each angle must be positive. We have the additional condition for acute triangles that

$$x, y, z \quad < \quad \frac{\pi}{2}$$

An effective way to approach this system is to fix z as z_0, and then consider the equation $x + y = \pi - z_0$. This projects the three-dimensional system into the XY-plane. We can now graphically represent this situation in the XY-plane as follows:

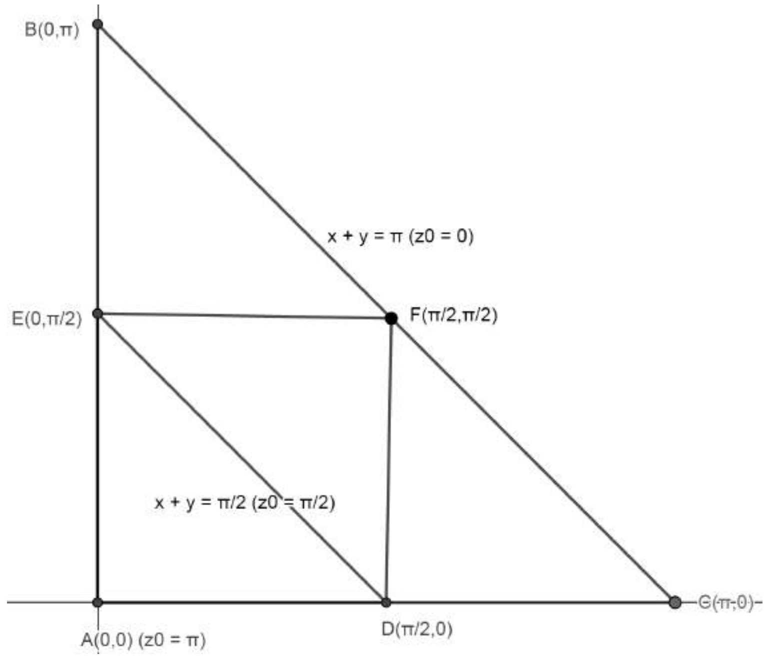


Figure 8.1

Notice that each point in the interior of the triangle ABC contains all the triplets (x, y, z) corresponding to a triangle, as the interior corresponds to $x + y < \pi$ and thus a suitable z may be chosen such that $x + y + z = \pi$. Meanwhile, each point on the boundary of ABC cannot construct a triangle as one or two angles must be zero; and any point outside ABC does not satisfy the interior angle sum property.

Also, observe that the interior of the triangle DEF contains all the triplets (x, y, z) corresponding to an acute triangle. Notice that each point on the boundary of DEF corresponds to one of x, y, z equalling $\frac{\pi}{2}$. And any point outside DEF corresponds to one of the angles exceeding $\frac{\pi}{2}$: $x > \frac{\pi}{2}$ in region CDF; $y > \frac{\pi}{2}$ in region BEF; and $z > \frac{\pi}{2}$ in region AED. Thus, the probability that a given triangle is acute is:

$$Pr(x < \frac{\pi}{2} \mid y < \frac{\pi}{2} \mid z < \frac{\pi}{2}) = \frac{Ar\triangle DEF}{Ar\triangle ABC} = \frac{\frac{1}{2}(\frac{\pi}{2})(\frac{\pi}{2})}{\frac{1}{2}(\pi)(\pi)} = \frac{1}{4}$$

It is seen that given a triangle, the probability that it is acute is $\frac{1}{4}$. Additionally, the triplets corresponding to right triangles lie on the lines DE, EF and FD - as they correspond to $z = \frac{\pi}{2}, y = \frac{\pi}{2}$ and $x = \frac{\pi}{2}$, respectively. Since lines have zero area, the probability of a random triangle being right-angled is zero! It also follows now that the proportion of all triangles which are obtuse is $\frac{3}{4}$.

BIBLIOGRAPHY

[1] Are random triangles acute or obtuse? - MIT Blossoms

THE THREE UTILITY PROBLEM

Apurva Chauhan
Year II

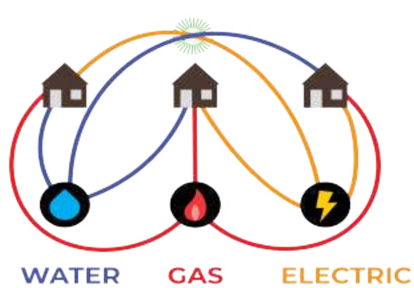
Abstract

Henry Ernest Dudeney was a mathematician who specialised in mathematical puzzles. He once stated that the problem discussed in this paper is "*as old as the hills ...much older than electric lighting, or even gas*". The utility problem has been a part of many magazines. In this article we will try to find a possible solution to our hoary problem using Graph Theory and the Jordan Curve Theorem. We'll also solve the classical problem using one of the most fascinating formulas of mathematics, given by Leonhard Euler.

Introduction

The utility problem is a sixty-four-thousand-dollar question. It is often referred to as a "*very ancient*" problem. This problem is one of the most famous maths puzzles and many mathematicians have tried to find an answer to this brainstorming puzzle. The problem is quoted as:

"*We have three houses and three utilities that are gas, water, and electricity and we need to connect each house to all three utilities by drawing lines on a paper without any line crossing each other as shown below:*"



First, we need to have a basic idea of Graph Theory, Euler's formula, and the Jordan Curve Theorem, because that is what we are going to use to find a possible solution to our problem.

Graph Theory

Graph theory is the study of connectivity between points called nodes (or vertices) which are connected through edges (links or lines). So, we will treat our houses and supplies as vertices and we will connect them using lines called edges without any point of intersection between them. In other words, we want a planar graph. A graph is a planar graph if it can be drawn in such a way that there is no edge intersection[5].

Theorem. *Any simply closed curve in the plane partitions the rest of the plane into two disjoint arc-wise connected open sets, one inside and one outside. It means that a loop will have an inside and an outside no matter how much we widen our lines, as long as the lines don't intersect[1].*

This is called the Jordan Curve Theorem.

Euler's Formula for Planar Graphs

It states that if a planar graph is drawn without any edge intersection then,

$$V - E + F = 2$$

Where:
V=number of vertices
E=number of edges
F=number of faces

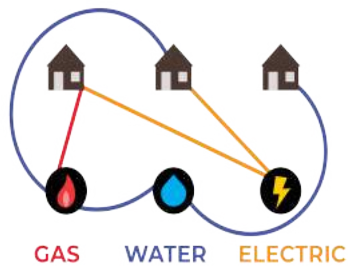
[2]

Solution

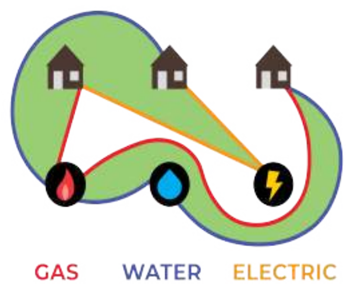
We will be trying to solve our problem through two possible ways using graph theory-

1. **Using The Jordan Curve Theorem** (see 9)

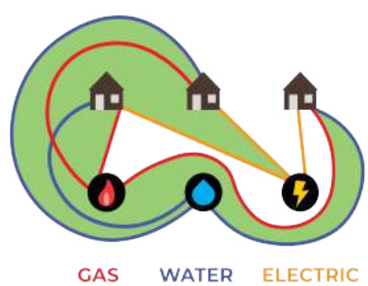
We will begin by joining some utilities to some of our houses. We joined the leftmost house to gas and electricity, middle house to water and electricity and rightmost house to water[3].



Now, if we draw one more line from gas utility to the rightmost house we will have a loop.



We can see that our loop has an inside and an outside. Now, we know that according to the Jordan Curve Theorem a loop will always have an inside and an outside until the lines don't cross. Now, to draw the remaining three lines we need to decide whether we want to draw these new lines inside or outside our loop. Joining the leftmost house to water will be easy but as we can see below for joining the other 2 lines we would definitely have to cross.



Therefore, from Jordan Curve Theorem we have shown that it is impossible to draw them without any line interaction.

2. **Using Euler's Formula for Planar Graph**

We will first use Euler's formula to find the number of faces a solution to our utility problem must have.

We know that;
V= number of vertices=6,
E=number of edges=9,
Using Euler's formula, we have
F=number of faces

$$F = 2 + E - V$$

$$F = 2 + 9 - 6 = 5$$

So, we have *five faces* according to our Euler's formula for planar graphs. We know that we can never connect 2 houses or 2 supplies together, that is we can never have 3 edges to a face. The problem states that connecting lines can only be drawn between houses and utilities. Which means that each face will have at least 4 edges. This implies that 5 faces will have at least 20 edges (5×4). But this counts each edge twice because every edge is a boundary for 2 faces. Therefore, we will take the smallest number of edges[4].

$$E = \frac{20}{2} = 10$$

But, according to the problem we can have only 9 edges!

Conclusions

The solution to the three utility problem is 'impossible'. It is impossible to connect the houses in the Euclidean plane (a flat sheet stretched to infinity).

BIBLIOGRAPHY

[1] Dhruv Gowswami-The Three Utilities Problem

[2] Euler's Formula and the Utilities Problem

[3] Math Forum-Three Utilities Problem

[4] Wikipedia-Graph Theory

[5] Wikipedia-Three Utilities problem

THE LOGARITHMIC SPIRAL

Khushi Agarwal
Year I

Abstract

This article is an attempt to provide some basic information about the famous logarithmic spiral. It includes the mathematical representation of the logarithmic spiral, its existence in nature, and a few of its real-world applications.

Introduction

We all are familiar with the logarithmic function and the ease with which it simplifies the otherwise time-consuming calculations. The logarithmic spiral was initially described by *Descartes*, but soon it grabbed the interests of other mathematicians like *Jacob Bernoulli*, who named it *"spira mirabilis"* or *"the marvelous spiral"*[1]. The properties of the logarithmic spiral fascinated Jacob to such an extent that he wanted it to be engraved on his tombstone along with the sentence, "*Although changed, I shall rise the same*", inscribed on it. But unfortunately, the artisan carved an *Archimedes'* spiral instead of the logarithmic one on his tombstone.

The Logarithmic Spiral

The logarithmic spiral is a *self-similar curve* in which the size of the curve increases with every successive turn, but its shape remains the same. The *polar equation*[1] of the logarithmic spiral can be given as:

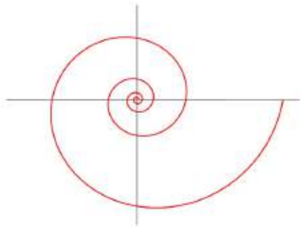
$$r = ae^{b\varphi}$$

where φ is a real number with $a > 0$ and $b \neq 0$ being real constants, and r and φ are the polar coordinates of the curve (r is the distance of any point on the spiral from the origin).

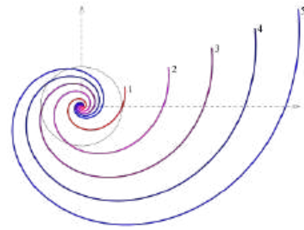
Cartesian coordinates: The cartesian representation of the above spiral is written as:

$x = ae^{b\varphi} \cos \varphi$, $y = ae^{b\varphi} \sin \varphi$

[1]



(a) A logarithmic spiral[6]



(b) Examples for $a = 1, 2, 3, 4, 5$ [7]

Properties of the Logarithmic Spiral

- The rate at which r increases or decreases is given by the constant a. The value of r increases in the counter-clockwise sense if $a > 0$, forming a left-handed spiral, and the value of r decreases in the counter-clockwise sense if $a < 0$, forming a right-handed spiral[4].
- With an increase in φ, the distance from the origin r increases in geometric progression[4].
- It is also known as the *equiangular spiral* because the angle between the radius drawn to a point from the center of the spiral and the tangent at that point remains constant throughout.
- If we come towards the center of the spiral from any point on it, an infinite number of rotations would be needed, but the distance covered in doing so will be finite[6].

How to construct a Logarithmic Spiral?

Logarithmic spirals can be constructed using golden triangles[3]. A *golden triangle* is an isosceles triangle in which the ratio of the length of equal sides to the length of the third side is equal to the golden ratio, 1.618034.

If we bisect one of the base angles of a golden triangle, another golden triangle can be generated. This procedure can be continued an infinite number of times to create smaller and smaller triangles. The vertices of these triangles can be joined to create a logarithmic spiral.

Logarithmic Spiral in Nature

It is quite fascinating that there exist many curves in several natural phenomena that roughly resemble the logarithmic spiral. Some of them are as follows:

- The chambers inside the shell of a nautilus are arranged in a rough logarithmic spiral[5].
- An approximate logarithmic spiral can be observed in the arms of a cyclone[1].

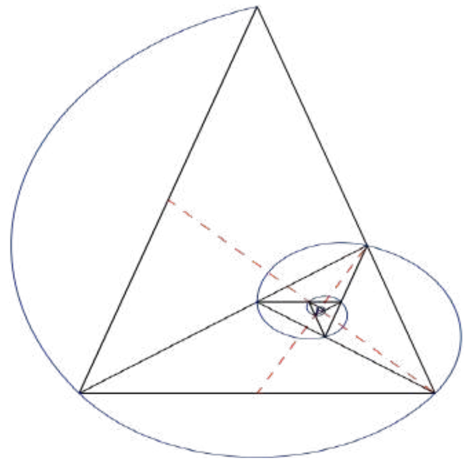


Figure 10.2: Golden Triangles inscribed in a Logarithmic Spiral[9]

- A special kind of broccoli, Romanesco broccoli grows in a pattern similar to the logarithmic spiral.
- Logarithmic spiral-like pattern can also be observed in the arms of spiral galaxies. The Milky Way galaxy has many spiral arms that resembles logarithmic spirals[1].

(a) Spiral shell of nautilus[10]

(b) Romanesco broccoli[11]

Figure 10.4: Arms of spiral galaxy

Conclusions

There are several other spirals like the *Archimedean spiral*, *Euler spiral*, *Fibonacci spiral*, etc. The logarithmic spiral is as fascinating as any one of these. It is used to make spiral bevel gears that are known for their excellent engineering characteristics[1]. It is also used in frequency-independent spiral antennas. Also, the ways in which it exists in nature are mind-blowing.

BIBLIOGRAPHY

[1] Wikipedia- Logarithmic spiral

[2] Logarithmic Spiral: Wolfram Math World

[3] Wikipedia- Golden triangle (mathematics)

[4] Article. Logarithmic Spiral - A Spiral Curve: Utpal Mukhopadhyay

[5] Wikipedia- Logarithm

[6] Logarithmic spiral: David Darling

[7] Image via Wolfram Math World

[8] Image via Wikipedia/Spiral Log

[9] Image via Wikipedia/Fibonacci Spiral

[10] Image via Shallow Sky

[11] Image by Jon Sullivan via Math Images

MATHEMATICS IN PLANTS

Priyanka Chopra
Year II

Abstract

This paper is an attempt to understand how mathematics is applied in plants. Complex calculations happening in plants every day can be well-explained by the mathematical concepts of Fibonacci Series and Golden Ratio. It focuses on the Fibonacci Series in particular. The paper discusses few of the applications of how mathematics is applied in various plants in brevity.

Keywords: *Fibonacci Series, Golden Ratio*

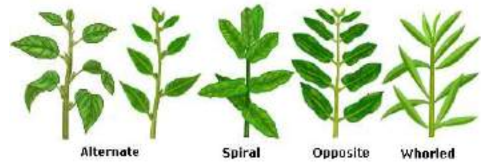
Introduction

Most of us are unaware of the fact that plants apply mathematics every single day to ensure that they have ample amount of resources to get through nights when sunlight isn't available. According to a recent analysis, plants cannot grow and thrive without mathematics. From arrangement of leaves, shape, size, and structure of plants to preparing food via photosynthesis, plants perform complex arithmetic calculations regularly. Formations and patterns in plants are not accidental, rather there is a mathematical logic behind these structures. It is commendable how symmetrical patterns can be observed in the arrangement of leaves and modeled using DC2 equation which can generate most leaf patterns.

Moreover, it is astonishing how the Fibonacci Series is widely used in predicting the position of the whorled leaves on stem. The Fibonacci Series can be used not only in studying the spiral pattern of seeds in sunflowers, but also in calculating the number of petals of various flowers.

While studying the structure of various plants, we get to realise that there is a well-ordered arrangement of leaves. This uniform arrangement of leaves around the stem is termed as *phyllotaxis*. Usually, the characteristics of this uniform pattern are explained in terms of phyllotactic patterns, including distichous, decussate, tricussate and Fibonacci patterns where spirally distributed tend to be separated by an angle of 137.5 degrees, also termed as the *Golden Angle*, which is the radial equivalent of the golden ratio.

(a)



(b)

Figure 11.1: Patterns in leaves[4]

It is essential to understand the *golden ratio* of the Fibonacci Series to unlock the mystery of these leaf patterns. Basically, in Fibonacci Series we take the sum of two preceding numbers to calculate the successive number i.e., the series goes like 1, 1, 2, 3, 5, 8, 13, ... To calculate the golden ratio, the ratio of two successive numbers in Fibonacci Series is taken and each number is divided by the preceding number as follows:

$$\frac{1}{1} = 1, \frac{2}{1} = 2, \frac{3}{2} = 1.5, \frac{5}{3} = 1.666\ldots, \frac{8}{5} = 1.6, \frac{13}{8} = 1.625, \frac{21}{13} = 1.61538\ldots$$

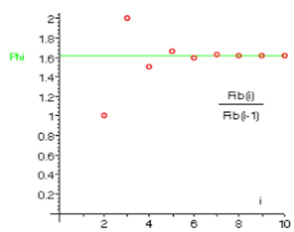


Figure 11.2: Graph Plotting of Fibonacci ratios

The ratio appears to be settling down to a particular value, which we call the *golden ratio* or the *golden number*. It has a value of 1.618034 approximately. Actually, the exact value of the golden ratio comes out to be Φ. As we proceed towards larger numbers in fibonacci series, the ratio turns out to be more close to Φ which can be explained by the basic Fibonacci Relationship:

$$F(i+2) = F(i+1) + F(i)$$

Therefore, as the series proceeds, the ratios get closer and closer to this limiting value Φ.

History of Fibonacci Series

Fibonacci Series is a significant contribution to mathematics by *Sir Leonardo Pisano Bigollo*, popularly known as '*Leonardo of Pisa*'. This series turned out to be an outcome of an arithmetic problem about Rabbit Breeding that was posed in the *Liber Abaci*. The problem which led to the discovery of Fibonacci Series was that starting with a pair of rabbits (1 male and 1 female), how many pairs of rabbits will be born in a year, assuming that every month each pair gives birth to a new pair of rabbits, and the new pair of rabbits itself starts giving birth to additional pairs of rabbits after the first month of their birth. Sir Leonardo ended up with what we study today as Fibonacci Sequence while searching the solution for the posed problem.

Later, it was *Kepler* who noted that in many different kinds of plants and trees, the leaves are aligned in a pattern that includes two Fibonacci numbers. Starting from any of the leaves, after one, two, three or five rotations of the spiral there is always a leaf aligned with the first and, depending on the species, this will be the second, the third, the fifth, the eighth, or the thirteenth leaf. This way Fibonacci became crucial not only in determining the structure pattern but also counting the number of petals.

Pattern of Various Flowers Using Fibonacci

Having a glimpse of some of the flowers in which the number of petals are in correspondence with Fibonacci results and a definite pattern can be identified using the series.

(a) White Calla Lily-1-petal flower [8]

(b) Iris-3-petal flower [6]

(c) Wild Rose-5-petal flower [7]

(d) Bloodroot-8-petal flower [5]

(e) Ragwort-13-petal flower [9]

Conclusions

It is confounding how the number of petals can be calculated using Fibonacci series and arithmetic calculations. It is perspicuous that there is a symmetrical pattern that is present in plants both in leaves and flowers. Mathematics can help us remarkably in finding the reasons of existing things in nature. The pattern, size of leaves, number of petals, and presence of recurrent structure in plants are not accidental, rather they have reliable premises which justify the way they are.

BIBLIOGRAPHY

[1] Shelley Allen, MASTER FIBONACCI *The Man Who Changed Math*, ISBN 978-15-3239-349-5

[2] https://fibonacci.com/nature-golden-ratio/

[3] Klar, A. 2002. Plant mathematics: Fibonacci's flowers. Nature. 417: 595

[4] Image by Maddie Burakoff via Smithsonian Magazine

[5] Image via Toronto Star

[6] Image via Kaggle

[7] Image by Jasmine Anenberg, Parks Conservancy

[8] Image via Michigan Bulb

[9] Photo taken at Mt. Cuba Center, accessed Mt.Cuba Center.org

PYRAMIDS AND MATHEMATICS

Mir Samreen
Year II

Abstract
This article is an attempt to show a real life application of mathematics associated with the Great Pyramids of Giza. In this article, Egyptian mathematics is briefly explained, followed by a reference to pyramids and how they encode several mathematical constants within their geometry and construction. The relation between Pi and Phi with the Great pyramids is discussed in order to understand how these Pyramids are interesting from a mathematical point of view.

Keywords: *Pi, Golden ratio, Fibonacci's numbers*

Introduction

The Great Pyramid of Giza is the oldest monument on the list of seven wonders of the world. It is a marvel of human engineering and construction. Till date, the biggest mystery about pyramids is the techniques used in their construction to erect them. It is a standard claim that the structure of the Great pyramid encodes a number of mathematical and physical constants. For example, it is quoted that

> "Dimensions of Great Pyramids contain the value of Pi, the principle of the golden section, Euler's number, the number of year in tropical year, the relative diameters of the earth at the equator and poles,..., the acceleration of gravity etc"[1].

Mathematics was an inductive discipline of utilitarian nature during the ancient Egyptian era. It was developed as a deductive science by the Greeks. Therefore, there is no evidence that Egyptian mathematicians used concept of textual geometry with construction and proofs. We get most of the information of Egyptian mathematics from the two major mathematical documents that have survived i.e. *Rhind and Moscow papyri*[2]. However, the mathematics involved in the Great Pyramid is so perfect that it is not limited to these two documents *per se*.

In the coming sections, connections between Pi, Phi, and Great Pyramids is discussed, since Pi and Phi are the two most important mathematical constants. It is fascinating to know how the connection between these constants and the pyramid might have been indented, but in a very different methodologies in a much more sophisticated manner.

Pi and the Great Pyramid

The perimeter of the base of the Great Pyramid divided by twice its height gives 3.150685, which is a remarkably accurate estimate of Pi. However, even though the pyramid gives a fair enough approximation of Pi, there is a reason to doubt that ancient Egyptians even had a concept of Pi, since the famous Rhind mathematical papyrus (written during the 13th dynasty) gives the value of Pi (3.160494), less accurate than the value of Pi obtained from pyramids erected in the 4th dynasty. But, there is another side to this conundrum. The relation between slope and Pi is given by the equation:

$$\tan 51.85^\circ = \frac{4}{\pi}$$

where 51.85° is the slope angle of the pyramid and the value of Pi obtained from this equation is 99.99 % accurate. One may look back at the system of measurements used by Babylonians, especially Egyptians, known as the method of *seked*. Seked is a measure of the slope where,

- 1 cubit = 7 palms
- 1 palm = 4 digits

The theory says that the Great Pyramid is based on the application of a gradient of 5.5 sekeds. This method of Egyptian mathematicians (based on Rhind papyri), and the ways in which they represented lengths and slopes give a very high probability that they might have chosen the angle of at least one of the pyramids to be approximately equal to the value $\frac{4}{\pi}$, which exhibits the amazing relationship between Pi and the pyramid. Due to this, one might believe that this relationship can be an accidental resultant of their mathematics.

The importance of Pi was intriguingly represented while seeking a solution to the most famous and intricate problem ever posed in history called "Squaring the Circle". Egyptians were also tackling this problem and it was only after the understanding of the nature of Pi, this problem found some lead. With an aid of simple geometry and instruments such as compass and ruler, it seeks to find area of a square equal to the area of a given circle. In the late 19^{th} century we came to know that such a task is impossible to perform as a consequence of the Lindemann-Weierstrass theorem which proves that Pi (π) is transcendental and not like an algebraic irrational number that is the root of any polynomial with rational coefficients[?]. But there exists an interesting view to squaring the circle exercise since it was believed to have symbolic meaning:

> Because the circle is an incommensurable figure based on π, it is impossible to draw a square more than approximately equal to it. Nevertheless, the Squaring of the Circle is of great importance to the geometer-cosmologist because for him the circle represents pure, un-manifest spirit-space, while the square represents the manifest and comprehensible world. When a near-equality is drawn between the circle and square, the infinite is able to express its dimensions or qualities through the finite[3].

Therefore, the accurate approximation gives in itself a sense of aesthetic satisfaction, and the amazing fact is, the *Great Pyramid squares the circle.* If a circle is drawn in such a manner that its centre lies in the centre of the base of the pyramid, with a radius equal to the height of the pyramid then we get the circumference of the circle equal to the perimeter of the square containing the pyramid. Such geometry approximates the solution of "squaring the circle" sough by ancient geometers.

Phi and Great Pyramid

When the slant height of the face of the pyramid is divided by half the length of its base, the answer is Phi, *Golden Ratio* $(1.61803\ldots)$. Phi is the only number which has a very unique property of its square equal to one more than itself i.e.

$$\phi + 1 = \phi^2.$$

By observing such property of phi one can apply the Pythagorean equation to this, for example in the *Golden Triangle.* One can construct the golden triangle (also called Kepler's triangle) with sides ϕ, 1 and $\sqrt{\phi}$ as shown in figure 12.1.

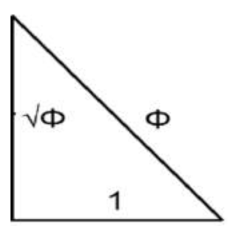


Figure 12.1: Golden Triangle

This can create pyramid constructed with the help of two such triangles placed back to back. Now, the ratio of height to the base is

$$\frac{\sqrt{\phi}}{1} = 0.636\ldots$$

An interesting computation is that the Great Pyramid has a base of 230.4 meters and an estimated (original) height of 146.5 meters. Now, we get

$$\frac{146.5}{230.4} = 0.636\ldots$$

This implies that Great Pyramid is also a golden triangle within three decimal places of accuracy. The pyramid based on the golden triangle has yet another interesting feature as well. For example, the surface area of the four sides of the pyramid will be the Golden Ratio of the surface area of its base. Calculation of the area of triangular sides can be given by $\frac{base \times height}{2}$ or $2 \times \frac{\phi}{2}$, $or\ \phi$. The surface area of the base is 4, so the four sides are $4 \times \frac{\phi}{4}$, $or\ \phi$ for the ratio of sides to base.

Another interesting, rather amazing information we get in accordance with phi's relation with pyramids comes from the relation of Golden Ratio (Phi) with the famous Fibonacci Sequence. The Fibonacci Sequence starts with 1 and 2. Every new number is the sum of the preceding two numbers.

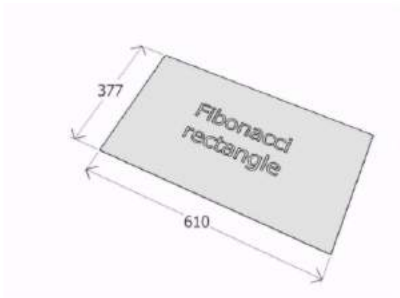
The Fibonacci numbers are: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$. Now, if we divide each number with previous number and keep repeating the process, we get

$1/1 = 1, 2/1 = 2, 3/2 = 1.5, \ldots, 144/89 = 1.6179\ldots$

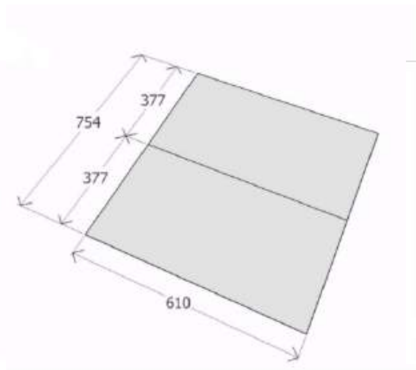
We observe that the successive ratio approaches 1.618, the golden ratio.

The mathematical astonishment is that, there is a Fibonacci's spiral that runs through the centre of three large Giza Pyramids, the three famous pyramids known i.e. *Pyramid of Khufu*, *Pyramid of Menkauru*, *Pyramid of Khafre*. This is indeed the fascinating real life application of the *Sacred Geometry*[4].

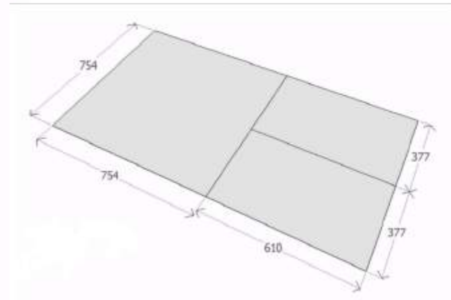
It is also said that the Great Pyramid of Giza is known as the Fibonacci Pyramid since it is constructed with Fibonacci numbers[5]. In the following page is a pictorial view of such construction.



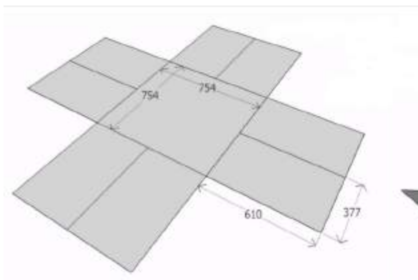
(a) A Fibinacci rectangle



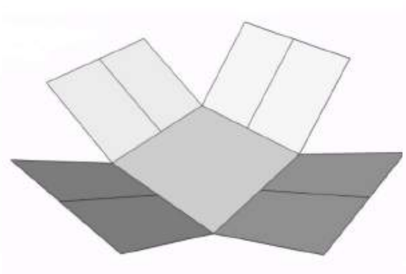
(b) Fibonacci rectangle copied to its side.



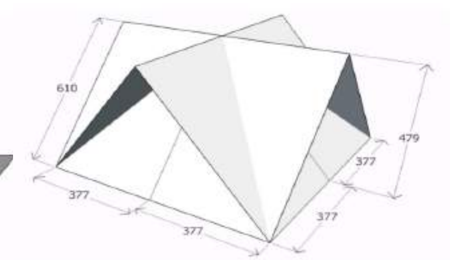
(c) A square is drawn



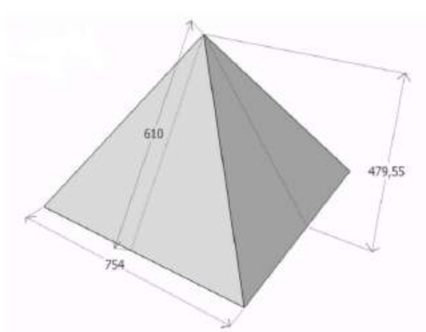
(d) Two Fibonacci rectangles are placed adjacent to each edge of square



(e) Rotating the rectangles



(f) Rotating till they meet in the middle



(g) Removing excess surfaces

Conclusions

The discussion regarding Pi, Phi, and the Great Pyramid of Giza gives an incredible real-life application of the beauty of mathematics. It is because history provides evidence that even if Egyptian mathematicians in ancient times used the numbers such as Pi and Phi, it's still a vague assumption that they even had any concept of these numbers in the decimal representation, since it was only after almost a century that the references of Phi were used by Greeks.

It then becomes feasible to believe as some do, that Egyptians might have used the integer approximation in their designs which resulted in the same relationships. Also, it is quite reasonable to believe this side as we know, both Pi and Phi (it's square root value) are such numbers that can be calculated approximately using simple integers that too with high degree precision.

$$\pi = \frac{22}{7} \text{ and } \phi = \frac{196}{121}$$

Also, one can also claim the possibility that perhaps Egyptians intended to use one of them in the dimensions of the pyramid and the other got automatically included. There are several questions regarding the construction techniques of pyramids from a mathematical perspective. No one is completely sure of how were pyramids actually designed. However, whatever is known one can say that the specification of a particular choice of the geometry for the Great Pyramid was doubtlessly done with some intention since it is different from the rest and truly perfect.

The Great Pyramid is one of the exquisite examples of the fact that beautiful, properly aligned, and coordinated things with nature require such a degree of mathematics in them.

BIBLIOGRAPHY

[1] Hunter Havelin Adams 3 in Portland *American-African Baseline Essays*

[2] Mathematics in Ancient Egypt (Part 1) by *Assem Deif, Cairo University, January 2007.*

[3] Robert Lawlor, Sacred Geometry Philosophy and Practice in *Sacred geometry of the Great Pyramid: squaring the circle, Skypal*

[4] The Great Pyramid

[5] The Fibonacci Pyramid of Giza by Ugur Gulcugil, *Erasmus University Rotterdam, May 2020.*

VERY SMALL PROOFS

Shivam Baurai
Year III

Abstract
In this article, 'clever'or 'simple'proofs of seemingly abstract problems are pursued, which are especially smaller than conventionally quoted proofs of the said results.

Wilson's Theorem

Lemma. *Let* $G = \{a_1, a_2, \ldots, a_n\}$ *be a finite abelian group and suppose that G has only one element of order 2,* y*. Then,* $\prod_{i=1}^{n} a_i = y$*.*

Theorem. *For a prime* p, $(p-1)! \equiv -1$ *mod*(p)

Proof. Consider the multiplicative group, $U(p)$.

$U(p) = \{n \in \mathbb{Z}_p \mid gcd(n, p) = 1\} = \{1, 2, 3, \ldots, p-1\}$ is cyclic. Thus, G has $\phi(2) = 1$ element of order 2, which is $p-1$.
Then, as per the lemma, $(p-1)! = (1)(2)\cdots(p-1) = p-1 \equiv -1$ mod(p)

□

The irrationality of $\sqrt{k}$

Theorem. $\sqrt{k}$ *is irrational* $\forall\, k \in \mathbb{N} : k$ *is not a perfect square.*

Proof. Assume otherwise. Then,$\sqrt{k}$ can be represented in lowest terms as: $\sqrt{k} = \frac{m}{n}$ where $m, n \in \mathbb{N}$ and $gcd(m, n) = 1$. Also note that since k is not a perfect square, $\exists\, j \in \mathbb{N} : j < \sqrt{k} < j+1$. Now:

1. $j < \sqrt{k} < j+1 \Rightarrow j < \frac{m}{n} < j+1 \Rightarrow nj < m < nj+n \Rightarrow 0 < m - nj < n$

2. Suppose,
$$\frac{kn - jm}{m - jn} < \frac{m}{n}.$$
Then,
$$kn^2 - jmn < m^2 - jmn \Rightarrow kn^2 < m^2 \Rightarrow k < (\frac{m}{n})^2 = k$$
which is a contradiction. Similar contradiction is reached if the inequality is reversed. Thus, it is concluded that
$$\frac{kn - jm}{m - jn} = \frac{m}{n} = \sqrt{k}$$

It is clear that $\frac{kn-jm}{m-jn}$ is an expression of $\sqrt{k}$ (from (2)) in terms lower than $\frac{m}{n}$ (from (1)), which is a contradiction.

Thus, $\sqrt{k}$ must be irrational. □

A THEOREM ON RATIONALITY OF EXPONENTIATION OF IRRATIONALS

Theorem. *There exist two irrationals a, b such that a^b is rational.*

Proof. Let $a = \sqrt{2}$ and $b = \sqrt{2}$.
Then, $a^b = \sqrt{2}^{\sqrt{2}}$.
If it is rational, we are done.
Otherwise, put $c = \sqrt{2}^{\sqrt{2}}$ and $d = \sqrt{2}$. Then,
$$c^d = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$$
which is rational. □

BIBLIOGRAPHY

[1] Topics in Algebra - I.N. Herstein

[2] A One-Sentence Proof That $\sqrt{2}$ Is Irrational - David M. Bloom (Brooklyn College of CUNY), Mathematics Magazine October 1995 (MAA)

[3] The Princeton Companion to Mathematics - T. Gowers

One-Step At A Time

ABSTRACT
Exploring the concept of defining a function between two non-empty sets and using it to solve a seemingly unrelated puzzle is the main objective of this article.

Keywords: *Function, Bijection*

What is a Bijection?

A function between two non-empty sets is said to be a bijection if it is both *injective* and *surjective*. Here, the domain and co-domain have intentionally been disguised because whenever there exists a bijection from a set to another, then an inverse map can also be defined the other way around.

Lets take a descriptive figure: Let f be a function $f : \mathcal{A} \to \mathcal{B}$, such that f is both *injective* and *surjective* then the mapping looks somewhat like this:

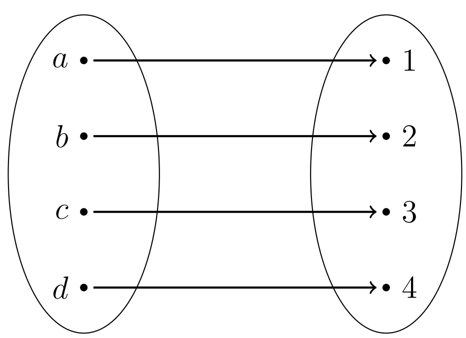
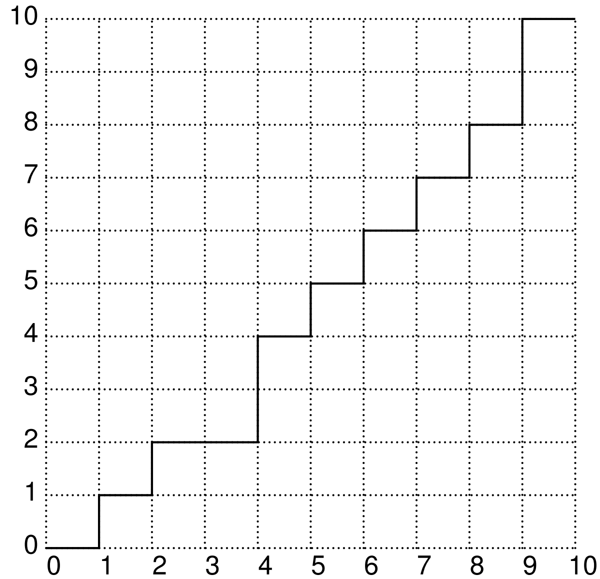


Figure 14.1

Problem: *Determine the number of walks from (0, 0) to (m, n) allowing only unit steps up or to the right.*
Solution: In this question, we try to morph the situation into something familiar. We try to symbolize each step and represent a right step by R and an up step by U. As an example, consider the following path:



According to our scheme we can define this path as under:

RURURRUURURURURURUUR

It can be seen that a similar path with m R's and n U's is possible, since the total possible R-steps are m and a maximum of n steps can be taken in the U-direction from (0,0) to (m,n). The key lies in the observation that we can construct a one-to-one correspondence between the above mentioned scheme of encryption and the set of paths from (0,0) to (m,n) using only unit up or right steps.

Now, counting the possibilities in the former case is as easy as counting the number of ways of placing m elements in m+n given spaces, which is, (C_m^{m+n}), and however simple it may seem now, this is the solution!

BIBLIOGRAPHY

[1] Shanzhen Gao, *The Enumeration of Lattice Paths and Walks*, 2011

[2] S.G. Mohanty, *Lattice Path Counting and Applications.* Academic Press, New York, 1979, ISBN 9781483205373

THE WINDMILL PROCESS

Abstract

We have reached a phase in this world where everyone fights an identity crisis. And people are appreciated for preserving their traditional and cultural ties. Though abstraction has made its own niche in the realm of art, any work of art is much appreciated and finds an even larger audience when it has a tinge of simplicity at its core. It is more of an irony that the questions that are genuinely tough have a cloak of intuitiveness attached to them as a disguise. In this paper, we attempt to discuss a seemingly simple problem that was asked in the world's toughest math test.

2011 IMO, Question 2

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times[1].

A Rough Sketch

The fact that a person of any age group or any academic background can understand the question makes it even harder to think about a mathematical proof. We begin by attempting to *mathematize* this question.

Firstly, to get a feel for what this *windmill* process is we begin by drawing some diagrams. After progressively increasing the number of points in $\mathcal{S}$, we observe that if the initial pivot of the line ℓ is an outside point then the line keeps on rotating around the perimeter of the plane and it never touches the interior points. But a second look at the question tells us that we have to find *some* starting point, not *any*! Then we notice that if we somehow start in the middle of the plane, then the line actually passes through each point and the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times. So the question boils down to how can we find this *middle-ness*?

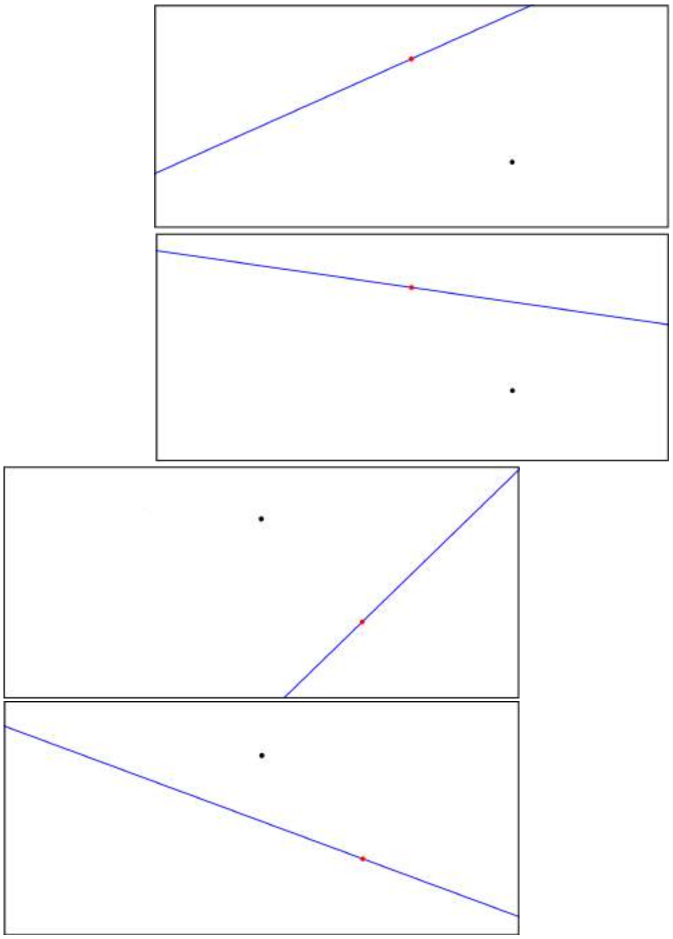


Figure 15.1: Case of 2 points

Solution 1

Starting from the point where we left, let us *math-o-fy* what we have observed so far. In a plane consisting of n points, we can say that the line ℓ is at the center, if the number of points on either of its sides are equal. Let us give this line an orientation so that we can distinguish its sides. To make this distinction even clearer, we call the line's left side the orange side and the other side is blue. It is easy to observe that whenever the pivot changes from a point A to a point B, A acquires the side initially possessed by B. Thus, the total elements of $\mathcal{S}$ on either of the sides remains constant throughout the *windmill* process (except for those moments when the line contains two points).

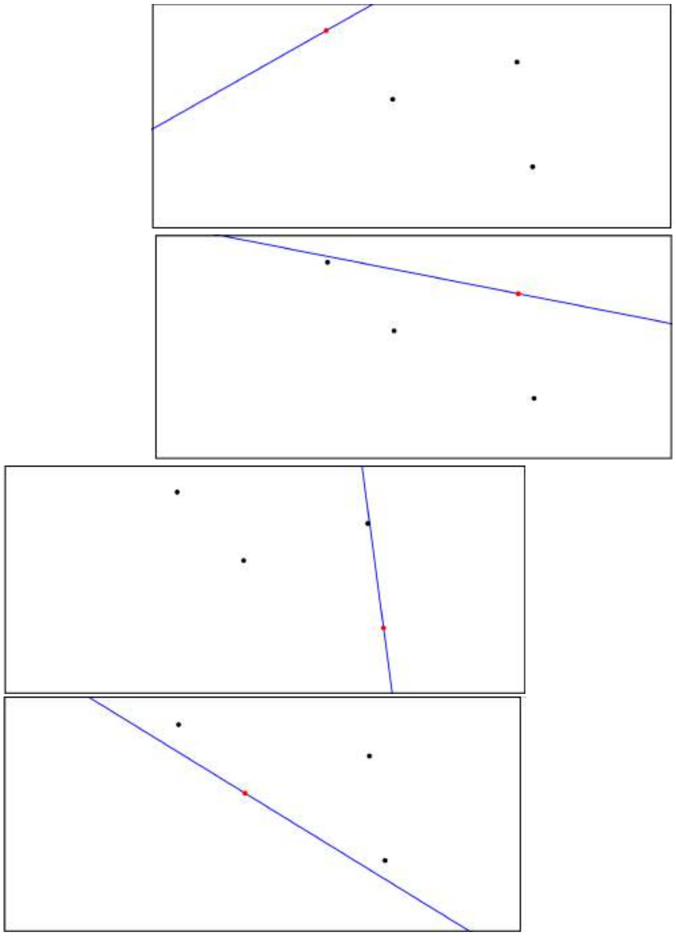
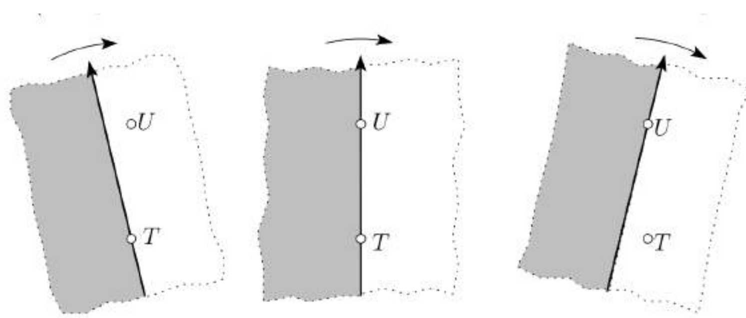


Figure 15.2: Case of 4 points



We consider two cases, in the first case let $|\mathcal{S}| = 2n + 1$ is odd. We claim that through any pivot $A \in \mathcal{S}$, there is a line that has n points on each side. To verify this, pick any oriented line through A that does not pass through any other point of $\mathcal{S}$ and suppose that it has $n+s$ points on its orange side. If $s = 0$ then we have established the claim, so we may assume that $s \neq 0$. As the line takes a half turn (180°) around A, the number of points of $\mathcal{S}$ on its orange side changes by 1 whenever the line passes through a point; after $180°$, the number of points on the orange side is $n - s$. Therefore, there is an intermediate stage at which the orange side, and thus also the blue side, contains n points.

Now, select the point B arbitrarily, and choose a line through B that has n points of $\mathcal{S}$ on each side to be the initial state of the windmill. We will show that during a rotation over $180°$, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point A of $\mathcal{S}$ and select a line ℓ through A that separates $\mathcal{S}$ into equal halves. The point A is the unique point of $\mathcal{S}$ through which a line in this direction can separate the points of $\mathcal{S}$ into equal halves (parallel translation would disturb the balance). Therefore, when the windmill line is parallel to ℓ, it must be ℓ itself, and so pass through A.

Next, suppose that $|\mathcal{S}| = 2n$. Similarly to the odd case, for every $A \in \mathcal{S}$ there is an oriented line through A with $n-1$ points on its orange side and n points on its blue side. Select such an oriented line through an arbitrary B to be the initial state of the windmill. We will now show that during a rotation over $360°$, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point A of $\mathcal{S}$ and an oriented line ℓ through A that separates $\mathcal{S}$ into two subsets with $n-1$ points on its orange and n points on its blue side. Again, parallel translation would change the numbers of points on the two sides, so when the windmill line is parallel to ℓ with the same orientation, the windmill line must pass through A. This completes the proof. □

Conclusions

This question definitely doesn't test the number of theorems that students have learned throughout their academic curriculum. This unusually pure puzzle has its charm in the intuitiveness attached to it that calls for a clever perspective.

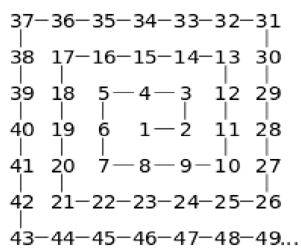
BIBLIOGRAPHY

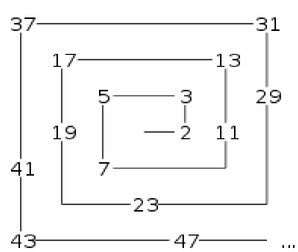
[1] 52nd International Mathematical Olympiad, 12-24 July 2011 Amsterdam, the Netherlands

MATHEMATICAL FACTS

Ulam Spiral

The Ulam Spiral or prime spiral is a plot of prime numbers, devised by mathematician *Stanislaw Ulam*. It is constructed by writing all the positive integers in a spiral arrangement on a square lattice and then by marking the primes[1].





From the spiral, it is clearly visible that most of the primes lie on diagonal straight lines and some on horizontal and vertical lines. Mostly, the number spiral is started with 1 at the centre but we can also start with any number of our choice. For example, if we start with 41 at the centre of spiral, we get a diagonal containing an unscathed string of 40 prime numbers (which is the longest example of its kind!).

Explanation. Diagonal, vertical and horizontal lines in the spiral corresponds to the polynomial of the form-

$$f(n) = 4n^2 + bn + c$$

where b and c are integer constants. If b is even, then the lines are diagonal, either all of odd or all even depending on the value of c.

However, some polynomials, *for example* $4n^2 + 8n + 3$ while producing all odd values, factorize to $(2n + 1)(2n + 3)$ and are therefore never prime other than 1. Such examples corrosponds to those diagonal lines that are lacking prime numbers.

HAILSTONE NUMBERS

The *Collatz Conjecture* is a conjecture that defines a sequence of positive integers in which each term is obtained from the previous term as follows:

- If n is even, the next term is $\frac{n}{2}$
- If n is odd, the next term is $3n + 1$

This conjecture was given by *Lothar Collatz.* It is also known as the $3n + 1$ *Problem* or $3n + 1$ Conjecture. According to this Conjecture, *regardless of the choice of n, eventually the sequences converges to 1.* The sequence of integers generated by this process are called *Hailstone Numbers*[2].

Examples

- If $n = 7$, we get the following sequence:

  $$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1$$

  Thus, when $n = 7$, it takes 17 steps to converge to 1.

- If $n = 27$, we get the following sequence:
  27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1

  When, $n = 27$, it takes 111 steps to converge to 1.

THE MOST BEAUTIFUL EQUATION

Euler's Identity is considered to be an epitome of mathematical beauty. The equality is

$$e^{i\pi} + 1 = 0$$

This equation is often compared to a Shakespearean sonnet or a Da Vinci picture. Euler's Identity encompasses the 3 basic arithmetic operations i.e. addition, multiplication and exponentiation. However, the main center of attraction is the fact that it comprises of the five most fundamental numbers in mathematics[3]:

1. $e =$ Euler's number, the base of natural logarithm
2. $i =$ Imaginary unit ($i^2 = -1$)
3. $\pi =$ Ratio of circumference of circle to its diameter
4. $1 =$ Multiplicative identity
5. $0 =$ Additive identity

WALLIS PRODUCT

The Wallis product for π is named after English mathematician *John Wallis*. It is as follows-

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \prod_{n=1}^{\infty} \frac{2n}{2n-1}\frac{2n}{2n+1} = \Big(\frac{2}{1}.\frac{2}{3}\Big)\Big(\frac{4}{3}.\frac{4}{5}\Big)\Big(\frac{8}{7}.\frac{8}{9}\Big)\cdots$$

[4]

Proof. We prove the above result using Euler infinite product for sine function, which is given by-

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \Big(1 - \frac{x^2}{n^2\pi^2}\Big)$$

Let $x = \frac{\pi}{2}$

$$\Rightarrow \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \prod_{n=1}^{\infty} \Big(1 - \frac{\frac{\pi^2}{4}}{n^2\pi^2}\Big)$$

$$\Rightarrow \frac{2}{\pi} = \prod_{n=1}^{\infty} \Big(1 - \frac{1}{4n^2}\Big)$$

$$\Rightarrow \frac{\pi}{2} = \prod_{n=1}^{\infty} \Big(\frac{4n^2}{4n^2-1}\Big) = \prod_{n=1}^{\infty} \Big(\frac{2n}{2n-1}\Big).\Big(\frac{2n}{2n+1}\Big)$$

□

The Four Colour Theorem

Have you ever got to colour a map for your geography project? How many colours did you use to colour it? In 1852, *Francis Guthrie* was trying to colour the map of counties of England. He noticed that *the regions could be coloured using at most four colours in such a way that no two adjacent regions had the same colour*[5]. The first successful proof of this conjecture came only in 1976, when *Kenneth Appel* and *Wolfgang Haken* proved it using computer assistance. They argued that if the four-colour conjecture were false, then there would be at least one map with minimum regions which needed five colours. Using two concepts they proved that such a minimal counterexample could not exist[6].

- Every non-4-colourable map must belong to an *unavoidable set* of configurations.
- If a configuration does not fall under minimal counterexample then it is called *reducible configuration*. A map can be reduced to a smaller map if it has a reducible configuration. If this smaller map can be coloured with four colours, so could the original one be. So, if the original map could not be coloured with four colours, the smaller one cannot either. Therefore, the original map is not minimal.

This proof remained contentious due to its extensive usage of computer assistance, but paved the path for its upcoming improvements.

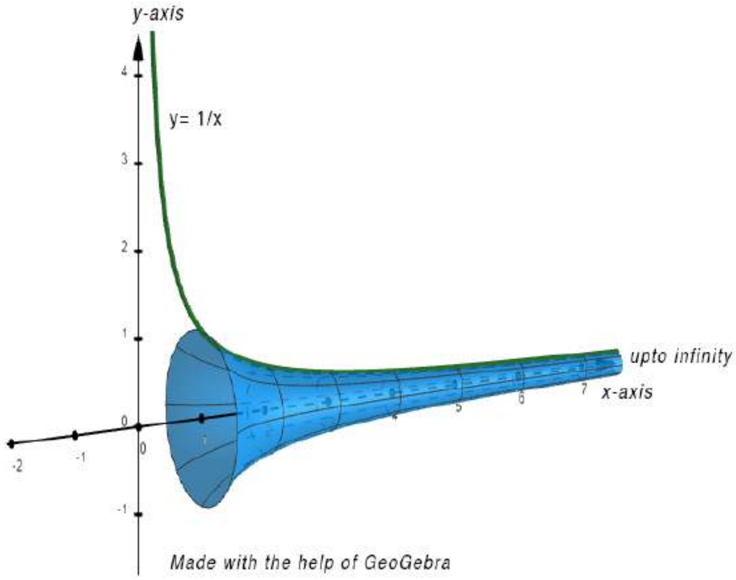
Gödel's Incompleteness Theorems

Do you think that mathematics is complete? Can all true statements be proved by a formal system (for example a system of natural numbers)? In 1931, *Kurt Gödel* came up with his two incompleteness theorems and showed that an axiomatic system cannot exist which contains all mathematics. It will only get bigger and bigger as we try to complete it[8].

- **First Incompleteness Theorem:** Given any consistent formal system *S*, it will be incomplete, that is, there exists statements in *S* which are neither provable nor disprovable in *S*.
- **Second Incompleteness Theorem:** Given any consistent formal system *S*, it is impossible to prove the consistency of *S* in *S* only[7].

Gabriel's Horn Paradox (Painter's Paradox)

Gabriel's Horn is the infinitesimally long surface of revolution obtained after rotating the curve $y = \frac{1}{x}$ (for $x \geq 1$) about the $x - axis$. Intuitively, when we think about its volume, it seems infinite as its length goes on to infinity and thus if we are to fill it with paint, it will always be somewhat empty. But, using calculus it turns out that it has a finite volume and infinite surface area, and thus the paradox[10].



Calculating Volume: Using disc method, the volume of the horn will be

$$\begin{aligned} V &= \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx \\ &= \pi \left[-\frac{1}{x}\right]_1^\infty \\ &= \pi \end{aligned}$$

Calculating Surface Area: The surface area of the horn will be:

$$\begin{aligned} S &= \int_1^\infty 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} dx \\ &= \int_1^\infty 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \frac{1}{x^4}} \; dx \\ &\geq \int_1^\infty 2\pi \, \frac{1}{x} \, dx \\ &= 2\pi \left[ln(x)\right]_1^\infty \end{aligned}$$

which is infinity.

Hence, the Gabriel's horn can be completely filled with π units of paint but cannot be painted fully as it has infinite surface area.

Relation with convergence and divergence of infinite series: The volume of Gabriel's horn is related to the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, that converges resulting in finite volume. Whereas the surface area is related to the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$, that diverges resulting in infinite surface area[9].

If one assumes the horn to be very thin, then the inner surface area would be same as the outer surface area. And if the horn can be completely filled with π units of paint, wouldn't the inner surface area automatically get painted? Again, the paradox!

BIBLIOGRAPHY

[1] Wikipedia-Ulam Spiral

[2] Wikipedia-Collatz Conjecture

[3] Euler's Identity

[4] Wallis Product

[5] The Four Colour Theorem

[6] FCT

[7] Incompletenss Theorems

[8] GIT

[9] Gabriel's Horn Paradox- Numberphile

[10] Wijeratne, Chanakya; Zazkis, Rina: On Painter's Paradox: Contextual and Mathematical Approaches to Infinity,International Journal of Research in Undergraduate Mathematics Education, July 2015 2198-9753